

# Three-dimensional $(p, q)$ AdS superspaces and matter couplings

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## Abstract

We introduce  $\mathcal{N}$ -extended  $(p, q)$  AdS superspaces in three space-time dimensions, with  $p + q = \mathcal{N}$  and  $p \geq q$ , and analyse their geometry. We show that all  $(p, q)$  AdS superspaces with  $X^{IJKL} = 0$  are conformally flat. Nonlinear  $\sigma$ -models with  $(p, q)$  AdS supersymmetry exist for  $p + q \leq 4$  (for  $\mathcal{N} > 4$  the target space geometries are highly restricted). Here we concentrate on studying off-shell  $\mathcal{N} = 3$  supersymmetric  $\sigma$ -models in  $\text{AdS}_3$ . For each of the cases  $(3,0)$  and  $(2,1)$ , we give three different realisations of the supersymmetric action. We show that  $(3,0)$  AdS supersymmetry requires the  $\sigma$ -model to be superconformal, and hence the corresponding target space is a hyperkähler cone. In the case of  $(2,1)$  AdS supersymmetry, the  $\sigma$ -model target space must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an  $\text{SO}(2)$  group of rotations of the two-sphere of complex structures.

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# 1 Introduction

In three space-time dimensions (3D), the anti-de Sitter (AdS) group is reducible,

$$\mathrm{SO}(2, 2) \cong \left( \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \right) / \mathbb{Z}_2 ,$$

and so are its supersymmetric extensions,  $\mathrm{OSp}(p|2; \mathbb{R}) \times \mathrm{OSp}(q|2; \mathbb{R})$ . This implies that  $\mathcal{N}$ -extended AdS supergravity exists in several incarnations [1]. These are known as the  $(p, q)$  AdS supergravity theories<sup>1</sup> where the non-negative integers  $p \geq q$  are such that  $\mathcal{N} = p + q$ . For arbitrary values of  $p$  and  $q$  allowed, the pure  $(p, q)$  AdS supergravity was constructed in [1] as a Chern-Simons theory with the gauge group  $\mathrm{OSp}(p|2; \mathbb{R}) \times \mathrm{OSp}(q|2; \mathbb{R})$ . Similar ideas can readily be used, e.g., to construct 3D higher-spin  $(p, q)$  AdS supergravity [3]. However, this Chern-Simons construction appears to become less powerful when it comes to coupling AdS supergravity to supersymmetric matter, especially in the important cases  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$ . In order to describe general supergravity-matter systems in these cases, superspace approaches appear to be the most useful ones.

As is well known, a universal approach to engineering supergravity theories in diverse dimensions is to realise them as conformal supergravity coupled to certain compensating

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<sup>1</sup>One should not confuse  $(p, q)$  AdS supersymmetry in three dimensions with  $(p, q)$  Poincaré supersymmetry in two dimensions [2].

supermultiplet(s) [4]. Making use of the conformal supergravity constraints on the superspace torsion proposed in [5], in our recent work [6] the superspace geometry of 3D  $\mathcal{N}$ -extended conformal supergravity was developed<sup>2</sup> and then applied (building on the structure of off-shell superconformal  $\sigma$ -models in three dimensions [12]) to construct general off-shell supergravity-matter couplings for the cases  $\mathcal{N} \leq 4$ . In order to illustrate how the formalism of [6] can be used to describe matter-coupled AdS supergravity theories, the cases  $p + q = 2$  were studied in detail in [13]. In particular, Ref. [13] provided two dual off-shell formulations for (1,1) AdS supergravity and one off-shell formulation for (2,0) AdS supergravity. The most general  $\sigma$ -model couplings to (1,1) and (2,0) AdS supergravity theories were constructed in [13] from first principles. These results generalise those obtained earlier [14, 15] within the Chern-Simons approach [1].<sup>3</sup>

The present paper is devoted to new applications of the formalism developed in [6]. First of all, here we introduce  $(p, q)$  AdS superspaces and study their geometric properties. Secondly, we develop an off-shell formalism for constructing rigid supersymmetric theories in AdS with  $p + q = 3$ , and specifically concentrate on describing general supersymmetric nonlinear  $\sigma$ -models.

Within the framework of [6],  $(p, q)$  AdS superspace

$$\text{AdS}_{(3|p,q)} = \frac{\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})}{\text{SL}(2, \mathbb{R}) \times \text{SO}(p) \times \text{SO}(q)} \quad (1.1)$$

originates as a maximally symmetric supergeometry with covariantly constant torsion and curvature generated by a symmetric torsion  $S^{IJ} = S^{JI}$ , with the structure-group indices  $I, J$  taking values from 1 to  $\mathcal{N}$ . It turns out that  $S^{IJ}$  is nonsingular, and the parameters  $p$  and  $q = \mathcal{N} - p$  determine its signature. The ordinary AdS space

$$\text{AdS}_3 = \frac{\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})}{\text{SL}(2, \mathbb{R})} \quad (1.2)$$

is the bosonic body of  $\text{AdS}_{(3|p,q)}$ . The curvature of  $\text{AdS}_3$  is proportional to  $S^2 = S^{IJ} S_{IJ} / \mathcal{N}$  (with the structure-group indices being raised and lowered using  $\delta^{IJ}$  and  $\delta_{IJ}$ ). The Killing

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<sup>2</sup>The special cases of  $\mathcal{N} = 8$  and  $\mathcal{N} = 16$  conformal supergravity theories were independently worked out in [7, 8] and [9] respectively. Recently, new results on  $\mathcal{N} = 8$  conformal supergravity and its applications have appeared [10, 11].

<sup>3</sup>Ref. [14] constructed only those  $\sigma$ -model couplings to (2,0) AdS supergravity in which the scalar fields are neutral under the gauged U(1)  $R$ -symmetry group. Ref. [15] studied locally supersymmetric  $\sigma$ -models on homogeneous spaces of the form  $G/H \times \text{U}(1)$  in which the scalar fields are charged under the gauged U(1)  $R$ -symmetry group. Such  $\sigma$ -model couplings to (2,0) AdS supergravity are special cases of those constructed in [13].

vector fields of  $\text{AdS}_{(3|p,q)}$  can be shown to generate the isometry group  $\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})$ . Among the superspaces  $\text{AdS}_{(3|p,q)}$  with  $p+q = \mathcal{N}$  fixed, the largest isometry group corresponds to  $\text{AdS}_{(3|\mathcal{N},0)} \equiv \text{AdS}^{3|2\mathcal{N}}$ .

In fact, starting from the superspace geometry of  $\mathcal{N}$ -extended conformal supergravity [6] and restricting the torsion to be covariantly constant and Lorentz invariant, a general AdS superspace solution for  $\mathcal{N} \geq 4$  includes not only the torsion  $S^{IJ}$  described above but also a completely antisymmetric torsion  $X^{IJKL} = X^{[IJKL]}$ . It turns out that the latter may be non-zero only if  $S^{IJ} = S\delta^{IJ}$ , which means  $p = \mathcal{N}$  and  $q = 0$ . Such solutions define new AdS superspaces,  $\text{AdS}_{S,X}^{3|2\mathcal{N}}$ , for which the isometry group is, in general, a subgroup of  $\text{OSp}(\mathcal{N}|2; \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ .

Why bother to study supersymmetric nonlinear  $\sigma$ -models in  $\text{AdS}_{(3|p,q)}$ ? Part of our motivation comes from four dimensions. Recently, it has been realised that rigid supersymmetric field theories in  $\text{AdS}_4$  have drastically different properties compared to their counterparts in Minkowski space [16, 17, 18, 19, 20, 21, 22]. Analogous results apply in five dimensions [23, 24]. It is therefore natural to study the specific features of rigid supersymmetric field theories in  $\text{AdS}_3$ .<sup>4</sup> And then we can immediately see that the 3D story is much richer than the 4D one, for in 3D there exist several versions of  $\mathcal{N}$ -extended AdS superspace. These superspaces have different isometry groups, and therefore they should allow different matter couplings.

This paper is organised as follows. In section 2 we introduce the  $(p, q)$  AdS superspaces and study their geometrical properties. In section 3 we prove that all  $(p, q)$  AdS superspaces with  $X^{IJKL} = 0$  are conformally flat. The specific features of the  $(p, q)$  AdS superspaces with  $p+q \leq 4$  are studied in section 4. In section 5 we develop a general setup to construct (3,0) and (2,1) supersymmetric theories in AdS. Specifically, we define a family of covariant projective supermultiplets to describe supersymmetric matter, and then present a manifestly supersymmetric action. We also give an expression for the action obtained by integrating out the superspace Grassmann variables. In section 6 we demonstrate how to reformulate any (3,0) and (2,1) supersymmetric field theory in AdS as a dynamical system in a certain  $\mathcal{N} = 2$  AdS superspace. In section 7 we construct general off-shell supersymmetric  $\sigma$ -models in AdS. Section 8 contains a brief discussion of the results obtained. Some details on the derivation of the component action (5.37) are collected in the appendix.

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<sup>4</sup>Locally supersymmetric nonlinear  $\sigma$ -models in three dimensions were constructed in the on-shell component approach in [25, 26].

## 2 Three-dimensional $(p, q)$ AdS superspaces

In this section, we develop the differential geometry of three-dimensional  $\mathcal{N}$ -extended  $(p, q)$  AdS superspaces.

### 2.1 Superspace geometry of $\mathcal{N}$ -extended conformal supergravity

All  $(p, q)$  AdS superspaces can be realised as special background configurations within the 3D  $\mathcal{N}$ -extended conformal supergravity that was originally sketched in [5] and then fully developed in [6]. In this subsection we recall those results of [6] which are necessary for our subsequent analysis.

In three dimensions,  $\mathcal{N}$ -extended conformal supergravity can be described using a curved superspace which is parametrized by real bosonic  $(x^m)$  and real fermionic  $(\theta_1^\mu)$  coordinates,

$$z^M = (x^m, \theta_1^\mu) , \quad m = 0, 1, 2 , \quad \mu = 1, 2 , \quad \mathbf{I} = \mathbf{1}, \dots, \mathcal{N} , \quad (2.1)$$

and is characterised by the structure group  $\text{SL}(2, \mathbb{R}) \times \text{SO}(\mathcal{N})$ . The superspace differential geometry is encoded in covariant derivatives of the form

$$\mathcal{D}_A \equiv (\mathcal{D}_a, \mathcal{D}_\alpha^I) = E_A + \Omega_A + \Phi_A , \quad (2.2)$$

where the tangent space indices take the values  $\alpha = 1, 2$ ,  $a = 0, 1, 2$ ,  $I = \mathbf{1}, \dots, \mathcal{N}$ . In eq. (2.2),  $E_A = E_A^M \partial_M$  is the supervielbein, with  $\partial_M = \partial / \partial z^M$ ;

$$\Omega_A = \frac{1}{2} \Omega_A^{bc} \mathcal{M}_{bc} = -\Omega_A^b \mathcal{M}_b = \frac{1}{2} \Omega_A^{\beta\gamma} \mathcal{M}_{\beta\gamma} , \quad \mathcal{M}_{ab} = -\mathcal{M}_{ba} , \quad \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha} \quad (2.3)$$

is the Lorentz connection; and

$$\Phi_A = \frac{1}{2} \Phi_A^{KL} \mathcal{N}_{KL} , \quad \mathcal{N}_{KL} = -\mathcal{N}_{LK} \quad (2.4)$$

is the  $\text{SO}(\mathcal{N})$ -connection. The Lorentz generators with two vector indices  $(\mathcal{M}_{ab})$ , with one vector index  $(\mathcal{M}_a)$  and with two spinor indices  $(\mathcal{M}_{\alpha\beta})$  are related to each other by the rules:  $\mathcal{M}_a = \frac{1}{2} \varepsilon_{abc} \mathcal{M}^{bc}$  and  $\mathcal{M}_{\alpha\beta} = (\gamma^a)_{\alpha\beta} \mathcal{M}_a$ . The generators of the group  $\text{SO}(\mathcal{N})$  are denoted by  $\mathcal{N}_{IJ}$ . For more details on our notation and conventions see Appendix A of [6]. The generators of  $\text{SL}(2, \mathbb{R}) \times \text{SO}(\mathcal{N})$  act on the covariant derivatives as follows:<sup>5</sup>

$$[\mathcal{M}_{ab}, \mathcal{D}_\alpha^I] = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_\alpha{}^\beta \mathcal{D}_\beta^I , \quad [\mathcal{M}_a, \mathcal{D}_\alpha^I] = -\frac{1}{2} (\gamma_a)_\alpha{}^\beta \mathcal{D}_\beta^I , \quad (2.5a)$$

$$[\mathcal{M}_{\alpha\beta}, \mathcal{D}_\gamma^I] = \varepsilon_{\gamma(\alpha} \mathcal{D}_{\beta)}^I , \quad [\mathcal{M}_{ab}, \mathcal{D}_c] = 2\eta_{c[a} \mathcal{D}_{b]} , \quad [\mathcal{M}_a, \mathcal{D}_b] = \varepsilon_{abc} \mathcal{D}^c , \quad (2.5b)$$

$$[\mathcal{N}_{KL}, \mathcal{D}_\alpha^I] = 2\delta_{[K}^I \mathcal{D}_{\alpha L]} , \quad [\mathcal{N}_{KL}, \mathcal{D}_a] = 0 . \quad (2.5c)$$

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<sup>5</sup>The operation of (anti) symmetrization of  $n$  indices is defined to involve a factor of  $(n!)^{-1}$ .

To describe conformal supergravity, the covariant derivatives have to obey certain constraints [5]. With the constraints imposed, the Bianchi identities lead to the following (anti) commutation relations<sup>6</sup> derived in [6]:

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 2i\varepsilon_{\alpha\beta}C^{\gamma\delta IJ}\mathcal{M}_{\gamma\delta} - 4iS^{IJ}\mathcal{M}_{\alpha\beta} \\ + \left(i\varepsilon_{\alpha\beta}X^{IJKL} - 4i\varepsilon_{\alpha\beta}S^{K[I}\delta^{J]L} + iC_{\alpha\beta}^{KL}\delta^{IJ} - 4iC_{\alpha\beta}^{K(I}\delta^{J)L}\right)\mathcal{N}_{KL} , \quad (2.6a)$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma^K] = -\left(\varepsilon_{\gamma(\alpha}C_{\beta)\delta}^{KL} + \varepsilon_{\delta(\alpha}C_{\beta)\gamma}^{KL} + 2\varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta}S^{KL}\right)\mathcal{D}_L^\delta \\ + \frac{1}{2}R_{\alpha\beta\gamma}^{Kde}\mathcal{M}_{de} + \frac{1}{2}R_{\alpha\beta\gamma}^{KPQ}\mathcal{N}_{PQ} . \quad (2.6b)$$

This algebra is given in terms of three dimension-1 tensor superfields,  $X^{IJKL}$ ,  $S^{IJ}$  and  $C_a^{IJ}$ , which are real and have the symmetry properties

$$X^{IJKL} = X^{[IJKL]} , \quad S^{IJ} = S^{(IJ)} , \quad C_a^{IJ} = C_a^{[IJ]} . \quad (2.7)$$

The dimension-3/2 components of the curvature in (2.6b),  $R_{\alpha\beta\gamma}^{Kde}$  and  $R_{\alpha\beta\gamma}^{KPQ}$ , are known algebraic functions [6] of first spinor covariant derivatives of the dimension-1 tensor superfields (2.7). It is useful to represent  $S^{IJ}$  as a sum of its irreducible components<sup>7</sup>

$$S^{IJ} = \mathcal{S}^{IJ} + \delta^{IJ}\mathcal{S} , \quad \delta_{KL}\mathcal{S}^{KL} = 0 , \quad \mathcal{S} := \frac{1}{\mathcal{N}}\delta_{KL}\mathcal{S}^{KL} . \quad (2.8)$$

The Bianchi identities imply the following set of differential equations [6]

$$\mathcal{D}_\alpha^I\mathcal{S}^{JK} = 2\mathcal{T}_\alpha^{I(JK)} + \mathcal{S}_\alpha^{(J}\delta^{K)I} - \frac{1}{\mathcal{N}}\mathcal{S}_\alpha^I\delta^{JK} , \quad (2.9a)$$

$$\mathcal{D}_\alpha^I C_{\beta\gamma}^{JK} = \frac{2}{3}\varepsilon_{\alpha(\beta}\left(C_{\gamma)}^{IJK} + 3\mathcal{T}_\gamma^{JKI} + 4(\mathcal{D}_\gamma^J\mathcal{S})\delta^{KI} + \frac{(\mathcal{N}-4)}{\mathcal{N}}\mathcal{S}_\gamma^{[J}\delta^{K]I}\right) \\ + C_{\alpha\beta\gamma}^{IJK} - 2C_{\alpha\beta\gamma}^{[J}\delta^{K]I} , \quad (2.9b)$$

$$\mathcal{D}_\alpha^I X^{JKLP} = X_\alpha^{IJKLP} - 4C_\alpha^{[JKL}\delta^{P]I} , \quad (2.9c)$$

where the dimension-3/2 superfields satisfy:  $\mathcal{T}_\alpha^{IJK} = \mathcal{T}_\alpha^{[IJ]K}$ ,  $\delta_{JK}\mathcal{T}_\alpha^{IJK} = \mathcal{T}_\alpha^{[IJK]} = 0$ ,  $C_{\alpha\beta\gamma}^{IJK} = C_{(\alpha\beta\gamma)}^{[IJK]}$ ,  $C_{\alpha\beta\gamma}^I = C_{(\alpha\beta\gamma)}^I$ ,  $C_\alpha^{IJK} = C_\alpha^{[IJK]}$ ,  $X_\alpha^{IJKPQ} = X_\alpha^{[IJKPQ]}$ .

The supergravity gauge group is generated by local transformations of the form

$$\delta_K\mathcal{D}_A = [K, \mathcal{D}_A] , \quad K = K^C\mathcal{D}_C + \frac{1}{2}K^{cd}\mathcal{M}_{cd} + \frac{1}{2}K^{PQ}\mathcal{N}_{PQ} , \quad (2.10)$$

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<sup>6</sup>For the purposes of this paper, we only need the explicit expressions for the dimension-1 components of the torsion and the curvature.

<sup>7</sup>In this paper, we make use of  $S^{IJ}$  and  $\mathcal{S}^{IJ}$ , as well as  $\mathcal{S}$  and  $S := \sqrt{\delta_{IK}\delta_{JL}S^{IJ}S^{KL}}/\mathcal{N}$ . We hope our imperfect notation will not lead to any confusion.

with all the gauge parameters obeying natural reality conditions but otherwise arbitrary. Given a tensor superfield  $T$ , it transforms as follows:

$$\delta_K T = K T . \quad (2.11)$$

The conformal supergravity constraints proposed in [5] are invariant under super-Weyl transformations. This invariance plays a key role in the discussion of the multiplet structure of  $\mathcal{N}$ -extended conformal supergravity in [5]. The super-Weyl transformations were not given explicitly in [5]. For  $\mathcal{N} = 8$  conformal supergravity, the finite form of super-Weyl transformations first appeared in [7]. In the case of  $\mathcal{N}$ -extended conformal supergravity, the infinitesimal form of these transformations was described in [6]. Here we present for the first time the finite form of the  $\mathcal{N}$ -extended super-Weyl transformations [6]. This result is essential for the analysis in section 3.

The super-Weyl transformation of the covariant derivatives is

$$\mathcal{D}'^I_\alpha = e^{\frac{1}{2}\sigma} \left( \mathcal{D}^I_\alpha + (\mathcal{D}^{\beta I} \sigma) \mathcal{M}_{\alpha\beta} + (\mathcal{D}_{\alpha J} \sigma) \mathcal{N}^{IJ} \right) , \quad (2.12a)$$

$$\begin{aligned} \mathcal{D}'_a = e^\sigma & \left( \mathcal{D}_a + \frac{i}{2} (\gamma_a)^{\alpha\beta} (\mathcal{D}_{(\alpha}^K \sigma) \mathcal{D}_{\beta)K} + \varepsilon_{abc} (\mathcal{D}^b \sigma) \mathcal{M}^c + \frac{i}{16} (\gamma_a)^{\alpha\beta} ([\mathcal{D}_{(\alpha}^{[K}, \mathcal{D}_{\beta)}^{L]} \sigma) \mathcal{N}_{KL} \right. \\ & \left. - \frac{i}{8} (\gamma_a)^{\alpha\beta} (\mathcal{D}_K^\rho \sigma) (\mathcal{D}_\rho^K \sigma) \mathcal{M}_{\alpha\beta} + \frac{3i}{8} (\gamma_a)^{\alpha\beta} (\mathcal{D}_{(\alpha}^{[K} \sigma) (\mathcal{D}_{\beta)}^{L]} \sigma) \mathcal{N}_{KL} \right) , \end{aligned} \quad (2.12b)$$

while the dimension-1 torsion and curvature tensors transform as

$$S'^{IJ} = e^\sigma \left( S^{IJ} - \frac{i}{8} ([\mathcal{D}^{\rho(I}, \mathcal{D}_{\rho}^{J)}] \sigma) + \frac{i}{4} (\mathcal{D}^{\rho(I} \sigma) (\mathcal{D}_{\rho}^{J)} \sigma) - \frac{i}{8} \delta^{IJ} (\mathcal{D}_K^\rho \sigma) (\mathcal{D}_\rho^K \sigma) \right) , \quad (2.13a)$$

$$C'_a{}^{IJ} = e^\sigma \left( C_a{}^{IJ} - \frac{i}{8} (\gamma_a)^{\alpha\beta} ([\mathcal{D}_{(\alpha}^{[I}, \mathcal{D}_{\beta)}^{J]} \sigma) - \frac{i}{4} (\gamma_a)^{\alpha\beta} (\mathcal{D}_{(\alpha}^{[I} \sigma) (\mathcal{D}_{\beta)}^{J]} \sigma) \right) , \quad (2.13b)$$

$$X'^{IJKL} = e^\sigma X^{IJKL} . \quad (2.13c)$$

For later use, we rewrite the super-Weyl transformations of  $S'^{IJ}$  and  $C'_a{}^{IJ}$  in the following equivalent form:

$$S'^{IJ} = \left( e^\sigma S^{IJ} - \frac{i}{4} (\mathcal{D}^{\rho(I} \mathcal{D}_{\rho}^{J)} e^\sigma) + \frac{i}{2} e^{-\sigma} \left( \delta_K^I \delta_L^J - \frac{1}{4} \delta^{IJ} \delta_{KL} \right) (\mathcal{D}^{\rho(K} e^\sigma) (\mathcal{D}_{\rho}^{L)} e^\sigma) \right) , \quad (2.14a)$$

$$C'_a{}^{IJ} = \left( C_a{}^{IJ} - \frac{i}{4} (\gamma_a)^{\alpha\beta} \mathcal{D}_{(\alpha}^{[I} \mathcal{D}_{\beta)}^{J]} e^\sigma \right) . \quad (2.14b)$$

This concludes our summary of the superspace geometry of  $\mathcal{N}$ -extended conformal supergravity [6].



## 2.2 Definition of $(p, q)$ AdS superspaces

We are now prepared to introduce AdS superspaces. By definition, they correspond to those conformal supergravity backgrounds which satisfy the following requirements:

- (i) the torsion and curvature tensors are Lorentz invariant;
- (ii) the torsion and curvature tensors are covariantly constant.

Condition (i) implies

$$C_a{}^{IJ} \equiv 0 . \quad (2.15)$$

Requirement (ii) has a series of implications. First of all, the conditions

$$\mathcal{D}_\alpha^I S^{JK} = \mathcal{D}_a S^{JK} = 0 , \quad \mathcal{D}_\alpha^I X^{JKLM} = \mathcal{D}_a X^{JKLM} = 0 \quad (2.16)$$

imply that all the dimension-3/2 curvatures in the second line of (2.6b) are identically zero. Moreover, the integrability conditions for the constraints (2.16),

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} S^{KL} = 0 , \quad \{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} X^{KLMN} = 0 , \quad (2.17)$$

are equivalent to the following algebraic constraints on  $S^{IJ}$  and  $X^{IJKL}$ :

$$0 = X^{IJN(K} S_N{}^{L)} - S^{IM} S_M{}^{(K} \delta^{L)J} + S^{JM} S_M{}^{(K} \delta^{L)I} , \quad (2.18a)$$

$$0 = X^{IJN[K} X_N{}^{LPQ]} + 2S^{M[I} \delta^{J]N} \delta_M^{[K} X_N{}^{LPQ]} - 2S^{M[I} \delta^{J]N} \delta_N^{[K} X_M{}^{LPQ]} . \quad (2.18b)$$

We now have to analyse all the implications of the Bianchi identities

$$\sum_{[ABC)} [\mathcal{D}_A, [\mathcal{D}_B, \mathcal{D}_C]] = 0 \quad (2.19)$$

in the case that the covariant derivatives obeying the (anti) commutation relations (2.6a)–(2.6b) are further subject to the constraints (i) and (ii). Solving the Bianchi identities is straightforward albeit somewhat tedious and we omit the details. By analysing (2.19) we obtain a new crucial constraint on the torsion tensor  $S^{IJ}$ :

$$\hat{S}^2 = S^2 \mathbb{1} , \quad \hat{S} := (S^{IJ}) = \hat{S}^T , \quad S^2 := \frac{1}{\mathcal{N}} \text{tr}(\hat{S}^2) \geq 0 . \quad (2.20)$$

This shows that  $\hat{S}$  is a nonsingular symmetric  $\mathcal{N} \times \mathcal{N}$  matrix if  $S^2 > 0$ ; in this case  $\hat{S}/S$  is an orthogonal matrix. Moreover, by solving the Bianchi identities one readily derives

a commutator of two vector covariant derivatives. The complete algebra of covariant derivatives turns out to be

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4iS^{IJ}\mathcal{M}_{\alpha\beta} + i\varepsilon_{\alpha\beta}\left(X^{IJKL} - 4S^{K[I}\delta^{J]L}\right)\mathcal{N}_{KL} , \quad (2.21a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S^J{}_K(\gamma_a)_\beta{}^\gamma\mathcal{D}_\gamma^K , \quad (2.21b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = 4S^2\varepsilon_{abc}\mathcal{M}^c = -4S^2\mathcal{M}_{ab} . \quad (2.21c)$$

We recall that the structure-group indices are raised and lowered using  $\delta^{IJ}$  and  $\delta_{IJ}$ . Due to (2.20), the constraints (2.18a)–(2.18b) become

$$0 = S^{(K}{}_N X^{L)IJN} , \quad (2.22a)$$

$$0 = X_N{}^{IJ[K} X^{LPQ]N} + S^{I[K} X^{LPQ]J} - S^{J[K} X^{LPQ]I} - S^{IM} X_M{}^{[LPQ}\delta^{K]J} + S^{JM} X_M{}^{[LPQ}\delta^{K]I} . \quad (2.22b)$$

In accordance with (2.20), there are two conceptually different cases: (a)  $S > 0$ ; and (b)  $S = 0$  and hence  $S^{IJ} = 0$ . In the former case, eq. (2.21c) tells us that the commutator of two vector covariant derivatives is exactly that of 3D AdS space. The latter case corresponds to a flat superspace with the following algebra of covariant derivatives:

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} + i\varepsilon_{\alpha\beta}X^{IJKL}\mathcal{N}_{KL} , \quad (2.23a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = 0 , \quad [\mathcal{D}_a, \mathcal{D}_b] = 0 . \quad (2.23b)$$

This superspace is of Minkowski type for  $\mathcal{N} = 1, 2, 3$ . However, for  $\mathcal{N} \geq 4$  there may exist a non-zero constant antisymmetric tensor  $X^{IJKL}$  constrained by

$$X_N{}^{IJ[K} X^{LPQ]N} = 0 , \quad (2.24)$$

so that the resulting superspace is a deformation of  $\mathcal{N}$ -extended Minkowski superspace. In what follows, our analysis will be restricted to the AdS case,  $S > 0$ .

## 2.3 Analysis of the $(p, q)$ AdS constraints

We have seen that the (anti) commutation relations (2.21) require the algebraic constraints (2.20) and (2.22) as the consistency conditions. Let us analyse the implications of these equations. The most important equation to study is (2.20).

The torsion  $\hat{S} = (S^{IJ})$  is a real symmetric  $\mathcal{N} \times \mathcal{N}$  matrix. A local  $\text{SO}(\mathcal{N})$  transformation can be performed to diagonalise  $\hat{S}$ . Then, without loss of generality, the general

solution of the constraint (2.20) is

$$S^{IJ} = S \operatorname{diag}(\overbrace{+1, \dots, +1}^p, \overbrace{-1, \dots, -1}^{q=\mathcal{N}-p}) , \quad (2.25)$$

where  $S = \sqrt{(S^{IJ}S_{IJ})/\mathcal{N}} > 0$  is a positive parameter of unit dimension. In the ‘diagonal frame’ (2.25), we are left with an unbroken local group  $\mathrm{SO}(p) \times \mathrm{SO}(q)$ . In what follows, we assume  $p \geq q$ . Such a superspace should originate as a maximally symmetric solution of the  $(p, q)$  AdS supergravity. The integers  $p$  and  $q$  determine the signature of  $S^{IJ}$ .

For our subsequent analysis, it is handy to introduce a special notation associated with the diagonal frame (2.25). All the isovector indices running from 1 to  $p$  will be overlined, while those taking values from  $p+1$  to  $\mathcal{N}$  will be underlined. With this notation, the components of  $S^{IJ}$  in the diagonal frame are

$$S^{\overline{I}\overline{J}} = S \delta^{\overline{I}\overline{J}} , \quad S^{\overline{I}\underline{J}} = -S \delta^{\overline{I}\underline{J}} , \quad S^{\underline{I}\overline{J}} = S^{\underline{I}\underline{J}} = 0 . \quad (2.26)$$

The diagonal frame is especially useful for solving the constraints obeyed by  $X^{IJKL}$ .

Making use of (2.26), it is easy to see that the constraint (2.22a) is equivalent to

$$X^{I\underline{J}\underline{K}\overline{L}} = 0 . \quad (2.27)$$

This means that the only non-zero components of  $X^{IJKL}$  are those which have all the indices of the same type, i.e.  $X^{\overline{I}\overline{J}\overline{K}\overline{L}}$  and  $X^{\underline{I}\underline{J}\underline{K}\underline{L}}$ .

Using (2.26) and (2.27), the second constraint on  $X^{IJKL}$ , eq. (2.22b), dramatically simplifies. The strongest condition arises when one chooses the index  $I$  overlined and the index  $J$  underlined. In this case, due to (2.26) and (2.27), the expression (2.22b) is non-trivial only if, among the indices  $K, L, P$  and  $Q$ , one is overlined and the other three are underlined or vice versa. We then get the following equations

$$0 = S X^{\underline{J}\underline{L}\underline{P}\underline{Q}} \delta^{\overline{K}\overline{I}} , \quad (2.28a)$$

$$0 = S X^{\overline{I}\overline{L}\overline{P}\overline{Q}} \delta^{\underline{K}\underline{J}} . \quad (2.28b)$$

It is clear that these equations can have nontrivial solutions only in the  $(\mathcal{N}, 0)$  case. We have thus proved that the curvature  $X^{IJKL}$  can only consistently appear in the AdS algebra if  $S^{IJ} = S \delta^{IJ}$ . In this case, the equation (2.22b) simplifies to

$$X_N^{IJ[K} X^{LPQ]N} = 0 . \quad (2.29)$$

This is the same algebraic equation which emerges in the case of deformed Minkowski superspace, eq. (2.24).

The first case where  $X^{IJKL}$  can appear in the algebra is  $\mathcal{N} = 4$ . Here

$$X^{IJKL} = X \varepsilon^{IJKL} , \quad (2.30)$$

where  $X$  is a real constant parameter, and  $\varepsilon^{IJKL}$  the completely antisymmetric Levi-Civita tensor (normalised by  $\varepsilon^{1234} = 1$ ) which is invariant under  $\text{SO}(4)$ . The constraint (2.29) is automatically satisfied due to the identity

$$\varepsilon_{IJKP} \varepsilon^{LMNP} = 6 \delta_{[I}^L \delta_J^M \delta_K^N] . \quad (2.31)$$

Here we do not give a general solution of eq. (2.29) for  $\mathcal{N} > 4$ . We just mention that a particular solution of eq. (2.29) for any  $\mathcal{N} > 4$  is obtained by choosing  $X^{IJKL} = X \varepsilon^{IJKL} \mathbf{56} \dots \mathcal{N}$ , with  $\varepsilon^{I_1 \dots I_{\mathcal{N}}}$  the appropriate Levi-Civita tensor. This solution is invariant under a subgroup  $\text{SO}(4) \times \text{SO}(\mathcal{N} - 4)$  of the gauged  $R$ -symmetry group  $\text{SO}(\mathcal{N})$ .

We conclude by rewriting the algebra of covariant derivatives (2.21a)–(2.21c) in the diagonal frame (2.25) for general  $(p, q)$  with  $q > 0$  (in the case  $(\mathcal{N}, 0)$  the algebra of covariant derivatives is given by eqs. (2.21a)–(2.21c) with  $S^{IJ} = S \delta^{IJ}$ )

$$\{\mathcal{D}_{\alpha}^{\bar{I}}, \mathcal{D}_{\beta}^{\bar{J}}\} = 2i \delta^{\bar{I}\bar{J}} \mathcal{D}_{\alpha\beta} - 4i S \delta^{\bar{I}\bar{J}} \mathcal{M}_{\alpha\beta} - 4i S \varepsilon_{\alpha\beta} \mathcal{N}^{\bar{I}\bar{J}} , \quad (2.32a)$$

$$\{\mathcal{D}_{\alpha}^I, \mathcal{D}_{\beta}^J\} = 2i \delta^{IJ} \mathcal{D}_{\alpha\beta} + 4i S \delta^{IJ} \mathcal{M}_{\alpha\beta} + 4i S \varepsilon_{\alpha\beta} \mathcal{N}^{IJ} , \quad (2.32b)$$

$$\{\mathcal{D}_{\alpha}^{\bar{I}}, \mathcal{D}_{\beta}^J\} = \{\mathcal{D}_{\alpha}^I, \mathcal{D}_{\beta}^{\bar{J}}\} = 0 , \quad (2.32c)$$

$$[\mathcal{D}_a, \mathcal{D}_{\beta}^{\bar{J}}] = S (\gamma_a)_{\beta}^{\gamma} \mathcal{D}_{\gamma}^{\bar{J}} , \quad [\mathcal{D}_a, \mathcal{D}_{\beta}^J] = -S (\gamma_a)_{\beta}^{\gamma} \mathcal{D}_{\gamma}^J , \quad (2.32d)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = 4 S^2 \varepsilon_{abc} \mathcal{M}^c = -4 S^2 \mathcal{M}_{ab} . \quad (2.32e)$$

Note that the  $R$ -symmetry group of this superspace is  $\text{SO}(p) \times \text{SO}(q)$ .

## 2.4 The Killing vector fields of $(p, q)$ AdS superspace

To describe rigid supersymmetric field theories in  $(p, q)$  AdS superspaces, we need to develop a superfield description of the corresponding isometry transformations. Here we use the diagonal frame where the results become more transparent, and consider only the cases  $X^{IJKL} = 0$ . The isometry transformations are generated by  $(p, q)$  AdS Killing vector fields,

$$\xi = \xi^a \mathcal{D}_a + \xi_I^{\alpha} \mathcal{D}_{\alpha}^{\bar{I}} + \xi_{\bar{I}}^{\alpha} \mathcal{D}_{\alpha}^I , \quad (2.33)$$

which by definition obey the equation

$$\left[ \xi + \frac{1}{2} \Lambda^{\bar{I}\bar{J}} \mathcal{N}_{\bar{I}\bar{J}} + \frac{1}{2} \Lambda^{IJ} \mathcal{N}_{IJ} + \frac{1}{2} \Lambda^{ab} \mathcal{M}_{ab}, \mathcal{D}_C \right] = 0 , \quad (2.34)$$

for some parameters  $\Lambda^{\bar{I}\bar{J}}$ ,  $\Lambda^{IJ}$  and  $\Lambda^{ab}$ . This equation is equivalent to the relations

$$\mathcal{D}_{\alpha}^{\bar{I}}\xi_{\beta}^{\bar{J}} = -\varepsilon_{\alpha\beta}\Lambda^{\bar{I}\bar{J}} + S\delta^{\bar{I}\bar{J}}\xi_{\alpha\beta} + \frac{1}{2}\delta^{\bar{I}\bar{J}}\Lambda_{\alpha\beta} , \quad \mathcal{D}_{\alpha}^I\xi_{\beta}^{\bar{J}} = 0 , \quad (2.35a)$$

$$\mathcal{D}_{\alpha}^I\xi_{\beta}^J = -\varepsilon_{\alpha\beta}\Lambda^{IJ} - S\delta^{IJ}\xi_{\alpha\beta} + \frac{1}{2}\delta^{IJ}\Lambda_{\alpha\beta} , \quad \mathcal{D}_{\alpha}^{\bar{I}}\xi_{\beta}^J = 0 , \quad (2.35b)$$

$$0 = \mathcal{D}_{\gamma\bar{I}}\xi^{\alpha\gamma} + 6i\xi_{\bar{I}}^{\alpha} , \quad 0 = \mathcal{D}_{\gamma I}\xi^{\alpha\gamma} + 6i\xi_I^{\alpha} , \quad (2.35c)$$

$$0 = \mathcal{D}_{\gamma}^{\bar{I}}\Lambda^{\alpha\gamma} + 12i S \xi^{\alpha\bar{I}} , \quad 0 = \mathcal{D}_{\gamma}^I\Lambda^{\alpha\gamma} - 12i S \xi^{\alpha I} , \quad (2.35d)$$

$$0 = \mathcal{D}_{(\alpha}^{\bar{I}}\xi_{\beta\gamma)} = \mathcal{D}_{(\alpha}^I\xi_{\beta\gamma)} = \mathcal{D}_{(\alpha}^{\bar{I}}\Lambda_{\beta\gamma)} = \mathcal{D}_{(\alpha}^I\Lambda_{\beta\gamma)} \quad (2.35e)$$

which imply the standard Killing vector equation

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = 0 . \quad (2.36)$$

In accordance with (2.35), the parameters  $\xi_{\bar{I}}^{\alpha}$ ,  $\xi_I^{\alpha}$ ,  $\Lambda^{\bar{I}\bar{J}}$ ,  $\Lambda^{IJ}$  and  $\Lambda^{ab}$  are uniquely determined in terms of  $\xi^a$ . It can be shown that the  $(p, q)$  AdS Killing vector fields generate the AdS supergroup  $\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})$ .

### 3 Conformal flatness of $(p, q)$ AdS superspaces

It was demonstrated in [13] that  $\text{AdS}_{(3|2,0)}$  and  $\text{AdS}_{(3|1,1)}$  are conformally flat superspaces. Here we generalise this result to the case of arbitrary  $(p, q)$  AdS superspaces with  $X^{IJKL} = 0$ . All superspaces  $\text{AdS}_{(3|p,q)}$  are demonstrated to be conformally flat. Since the super-Weyl transformation of  $X^{IJKL}$  is homogeneous, any AdS superspace with  $X^{IJKL} \neq 0$  is not conformally flat.

The super-Weyl transformations in  $\mathcal{N}$ -extended conformal supergravity are given by eqs. (2.12a)–(2.12b). Our goal is to show that there exists a local parametrisation of the superspace  $\text{AdS}_{(3|p,q)}$  such that the covariant derivatives  $\mathcal{D}_A$  take the form

$$\mathcal{D}_{\alpha}^I = e^{\frac{1}{2}\sigma} \left( D_{\alpha}^I + (D^{\beta I}\sigma)\mathcal{M}_{\alpha\beta} + (D_{\alpha J}\sigma)\mathcal{N}^{IJ} \right) , \quad (3.1a)$$

$$\begin{aligned} \mathcal{D}_a = e^{\sigma} \Big( \partial_a + \frac{i}{2}(\gamma_a)^{\alpha\beta}(D_{(\alpha}^K\sigma)D_{\beta)K} + \varepsilon_{abc}(\partial^b\sigma)\mathcal{M}^c + \frac{i}{16}(\gamma_a)^{\alpha\beta}([D_{(\alpha}^{[K}, D_{\beta)}^{L]}\sigma)\mathcal{N}_{KL} \\ - \frac{i}{8}(\gamma_a)^{\alpha\beta}(D_K^{\rho}\sigma)(D_{\rho}^K\sigma)\mathcal{M}_{\alpha\beta} + \frac{3i}{8}(\gamma_a)^{\alpha\beta}(D_{(\alpha}^{[K}\sigma)(D_{\beta)}^{L]}\sigma)\mathcal{N}_{KL} \Big) , \end{aligned} \quad (3.1b)$$

for some real scalar  $\sigma$ . Here  $D_A = (\partial_a, D_{\alpha}^I)$  are the covariant derivatives of  $\mathcal{N}$ -extended 3D Minkowski superspace,

$$\partial_a = \frac{\partial}{\partial x^a} , \quad D_{\alpha}^I = \frac{\partial}{\partial \theta_{\alpha}^I} + i(\gamma^a)_{\alpha\beta}\theta^{\beta I}\partial_a , \quad (3.2a)$$

obeying the (anti) commutation relations

$$\{D_\alpha^I, D_\beta^J\} = 2i\delta^{IJ}(\gamma^a)_{\alpha\beta}\partial_a, \quad [\partial_a, D_\beta^J] = [\partial_a, \partial_b] = 0. \quad (3.2b)$$

Under the super-Weyl transformations, the dimension-1 torsion and curvature superfields transform according to the equations (2.13a)–(2.13c). The superspace  $\text{AdS}_{(3|p,q)}$  is characterised by the conditions  $C_a^{KL} = X^{IJKL} = 0$ . Hence the parameter  $e^\sigma$  in (3.1) must satisfy the equations

$$S^{IJ} = -\frac{i}{4}(D^{\rho(I}D_\rho^{J)}e^\sigma) + \frac{i}{2}e^{-\sigma}(D^{\rho(I}e^\sigma)(D_\rho^{J)}e^\sigma) - \frac{i}{8}\delta^{IJ}e^{-\sigma}(D_K^\rho e^\sigma)(D_\rho^K e^\sigma), \quad (3.3a)$$

$$0 = D_{(\alpha}^{[I}D_{\beta]}^{J]}e^\sigma. \quad (3.3b)$$

Moreover, in accordance with the analysis of the previous section, the superfield  $S^{IJ}$  has to be covariantly constant,

$$\mathcal{D}_A S^{JK} = 0, \quad (3.4)$$

and obey the algebraic constraint (2.20) which we rewrite as

$$S^{IK}S_{KJ} = S^2\delta_J^I, \quad S^2 = \frac{1}{\mathcal{N}}S^{KL}S_{KL}. \quad (3.5)$$

The equations (3.4) and (3.5) have to be obeyed by  $e^\sigma$  in addition to the condition (3.3b).

To find a solution of the above equations, we make a Lorentz invariant ansatz for the super-Weyl parameter

$$e^\sigma = 1 + as^2x^2 - \Theta_s + bs^2\Theta^2 + c\Theta_s^2 + ds\Theta\Theta_s, \quad (3.6)$$

where

$$x^2 := x^a x_a, \quad \theta^{IJ} := \theta^I \theta_\gamma^J = \theta^{JI}, \quad \Theta := i\delta_{KL}\theta^{KL}, \quad \Theta_s := is_{KL}\theta^{KL}, \quad (3.7a)$$

$$s_{IJ} = s_{JI}, \quad s := \sqrt{\frac{s^{KL}s_{KL}}{\mathcal{N}}}. \quad (3.7b)$$

The constant parameters  $a, b, c$  and  $d$  in (3.6) are real and dimensionless. As to the constant tensor  $s_{IJ}$ , it is also real and has unit mass dimension.

The ansatz (3.6) has been shown to be the correct one in the case of the (2,0) and (1,1) AdS superspace [13]. It is also reminiscent of the conformally flat parametrisation of the 4D  $\mathcal{N}$ -extended AdS superspace derived in [27] using group-theoretic techniques (direct proofs based on the use of super-Weyl transformations in 4D  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  supergravity theories can be found, e.g., in [28] and [29] respectively).

Let us turn to solving the conditions for conformal flatness using the ansatz introduced. As a first step, we observe from eq. (3.3a) that

$$S^{IJ} = s^{IJ} + \mathcal{O}(\theta) . \quad (3.8)$$

By considering the  $\theta$ -independent part of (3.5), we see that  $s^{IJ}$  has to be constrained by

$$s^{IK} s_{KJ} = \delta_K^I s^2 . \quad (3.9)$$

Next, we impose the equation (3.3b). After some algebra, one sees that (3.3b) is satisfied provided

$$b = -\frac{a}{4} , \quad c = d = 0 . \quad (3.10)$$

To fix the value of  $a$ , it suffices to use again the equation (3.5) which so far has been solved at the  $\theta = 0$  order only. This equation tells us that

$$a = -1 . \quad (3.11)$$

We end up with the following expression for the super-Weyl parameter:

$$e^\sigma = 1 - s^2 x^2 - \Theta_s + \frac{1}{4} s^2 \Theta^2 = 1 - s^2 x^2 - i s^{KL} \theta_{KL} - \frac{1}{8} s^2 (\delta^{KL} \theta_{KL})^2 . \quad (3.12)$$

The geometry is characterized by the torsion  $S^{IJ}$  that, by using (3.3a), can be computed to be

$$S^{IJ} = s^{IJ} + 2i \frac{s^2 \theta^{IJ} - s^{K(I} s^{J)L} \theta_{KL} + 2s^2 s^{K(I} \theta_{\gamma}^{J)} \theta_{\delta K} x^{\gamma\delta} - s^2 \theta^{IJ} \Theta_s + s^2 s^{K(I} \theta^{J)}_{\phantom{J)}K} \Theta}{1 - s^2 x^2 - \Theta_s + \frac{1}{4} s^2 \Theta^2} . \quad (3.13)$$

It is an instructive exercise to prove the important relations

$$\mathcal{S} := \frac{1}{\mathcal{N}} \delta_{KL} S^{KL} = \frac{1}{\mathcal{N}} \delta_{KL} s^{KL} \quad \implies \quad \mathcal{D}_A \mathcal{S} = 0 , \quad (3.14a)$$

$$S^2 = s^2 \quad \implies \quad \mathcal{D}_A S^2 = 0 . \quad (3.14b)$$

To complete the analysis, we need to prove that the condition (3.4) holds, with the covariant derivatives defined by eqs. (3.1). This is equivalent to proving that  $\mathcal{D}_\alpha^I S^{JK} = 0$ . We can simplify such a check by making a series of simple considerations. First of all, due to the very nature of the super-Weyl transformations, the covariant derivatives (3.1) define some conformal supergravity background with the additional conditions:

the corresponding torsion and curvature tensors satisfy the equations (2.9a)–(2.9c) with  $X^{IJKL} = C_a^{KL} = 0$ . This means that the torsion  $S^{IJ} = \mathcal{S}^{IJ} + \delta^{IJ}\mathcal{S}$  in eq. (3.13) satisfies

$$\mathcal{D}_\alpha^I \mathcal{S}^{JK} = \mathcal{S}_\alpha^{(J} \delta^{K)I} - \frac{1}{\mathcal{N}} \mathcal{S}_\alpha^I \delta^{JK} , \quad (3.15a)$$

$$0 = 4\mathcal{N}(\mathcal{D}_\alpha^I \mathcal{S}) + (\mathcal{N} - 4)\mathcal{S}_\alpha^I . \quad (3.15b)$$

It follows that for  $\mathcal{N} \neq 4$ , a sufficient condition to have  $\mathcal{D}_\alpha^I \mathcal{S}^{JK} = 0$  is  $\mathcal{D}_\alpha^I \mathcal{S} = 0$ . This is indeed the case in accordance with (3.14a).

In the  $\mathcal{N} = 4$  case, the condition  $\mathcal{D}_\alpha^I \mathcal{S} = 0$  follows from (3.14a) and (3.15b). However we need to independently check whether the requirement  $\mathcal{S}_\alpha^I = 0$  holds indeed. Using  $\mathcal{D}_\alpha^I \mathcal{S} = 0$  and the representation  $S^2 = (\mathcal{S}^{KL} \mathcal{S}_{KL})/\mathcal{N} + \mathcal{S}^2$  gives

$$\mathcal{N} \mathcal{D}_\alpha^I \mathcal{S}^2 = 2(S^I{}_J \mathcal{S}_\alpha^J - \mathcal{S} \mathcal{S}_\alpha^I) . \quad (3.16)$$

Here the left-hand side is zero due to (3.14b), and hence

$$\mathcal{S} \mathcal{S}_\alpha^I = S^I{}_J \mathcal{S}_\alpha^J . \quad (3.17)$$

Since  $S^I{}_J$  is invertible, this equation gives  $\mathcal{S}_\alpha^I = 0$  in the case that  $\mathcal{S} = 0$ . On the other hand, choosing  $\mathcal{S} \neq 0$  in (3.17) gives

$$\mathcal{S}_\alpha^I = \left(1 + \frac{\mathcal{S}^{KL} \mathcal{S}_{KL}}{\mathcal{N} \mathcal{S}}\right) \mathcal{S}_\alpha^I . \quad (3.18)$$

Ultimately this equation tells us that  $\mathcal{S}_\alpha^I = 0$ . Therefore, we have demonstrated that  $S^{IJ}$  is covariantly constant.

We conclude with a comment about the space-time geometry associated with the superspace  $\text{AdS}_{(3|p,q)}$ . Given the expression for  $e^\sigma$ , eq. (3.12), and the explicit form of the vector covariant derivative  $\mathcal{D}_a$ , eq. (3.1b), we can read off the space-time metric

$$ds^2 = dx^a dx_a (e^{-2\sigma})|_{\theta=0} = \frac{dx^a dx_a}{(1 - s^2 x^2)^2} . \quad (3.19)$$

This coincides with a standard expression for the metric of  $\text{AdS}_3$  computed using the stereographic projection for an AdS hyperboloid.<sup>8</sup>

## 4 Elaborating on the AdS superspaces with $p + q \leq 4$

In this section we would like to reformulate the algebra of covariant derivatives, which corresponds to a given  $(p, q)$  AdS superspace with  $p + q \leq 4$ , in a form that is more suitable for describing matter couplings within the supergravity formulation of [6].

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<sup>8</sup>See, e.g., Appendix D of [29] for details about the stereographic projection for  $\text{AdS}_d$ .



## 4.1 $\mathcal{N} = 1$

In the  $\mathcal{N} = 1$  case, only the  $(1,0)$  AdS superspace is available. Its geometry is determined by the (anti) commutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\mathcal{D}_{\alpha\beta} - 4iS\mathcal{M}_{\alpha\beta} , \quad (4.1a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma , \quad (4.1b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = 4\varepsilon_{abc}S^2\mathcal{M}^c = -4S^2\mathcal{M}_{ab} . \quad (4.1c)$$

## 4.2 $\mathcal{N} = 2$

In the  $\mathcal{N} = 2$  case, there are two AdS superspaces:  $(2,0)$  and  $(1,1)$ . They have already been studied in [13]. Here we would like to re-derive the main results of [13] using the analysis of the previous section.

### 4.2.1 $(2,0)$ AdS superspace

The  $(2,0)$  covariant derivatives satisfy the (anti) commutation relations

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4iS\delta^{IJ}\mathcal{M}_{\alpha\beta} + 4\varepsilon_{\alpha\beta}S\varepsilon^{IJ}\mathcal{J} , \quad (4.2a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^J , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab} , \quad (4.2b)$$

where  $\varepsilon^{12} = \varepsilon_{12} = 1$  and we have introduced the  $U(1)$  generator  $\mathcal{J}$  following [6]

$$\mathcal{N}_{KL} = i\varepsilon_{KL}\mathcal{J} , \quad \mathcal{J} = -\frac{i}{2}\varepsilon^{PQ}\mathcal{N}_{PQ} , \quad [\mathcal{J}, \mathcal{D}_\alpha^I] = -i\varepsilon^{IJ}\mathcal{D}_{\alpha J} . \quad (4.3)$$

It is useful to switch to a complex basis for the spinor covariant derivatives,  $\mathcal{D}_\alpha^I \rightarrow (\mathbf{D}_\alpha, \bar{\mathbf{D}}_\alpha)$ , such that  $\mathbf{D}_\alpha$  and  $\bar{\mathbf{D}}_\alpha$  possess definite  $U(1)$  charges

$$\mathbf{D}_\alpha = \frac{1}{\sqrt{2}}(\mathcal{D}_\alpha^1 - i\mathcal{D}_\alpha^2) , \quad \bar{\mathbf{D}}_\alpha = -\frac{1}{\sqrt{2}}(\mathcal{D}_\alpha^1 + i\mathcal{D}_\alpha^2) , \quad (4.4a)$$

$$[\mathcal{J}, \mathbf{D}_\alpha] = \mathbf{D}_\alpha , \quad [\mathcal{J}, \bar{\mathbf{D}}_\alpha] = -\bar{\mathbf{D}}_\alpha . \quad (4.4b)$$

With the definition  $\mathbf{D}_a = \mathcal{D}_a$ , the algebra of covariant derivatives becomes

$$\{\mathbf{D}_\alpha, \mathbf{D}_\beta\} = 0 , \quad \{\mathbf{D}_\alpha, \bar{\mathbf{D}}_\beta\} = -2i\mathbf{D}_{\alpha\beta} - 4iS\varepsilon_{\alpha\beta}\mathcal{J} + 4iS\mathcal{M}_{\alpha\beta} , \quad (4.5a)$$

$$[\mathbf{D}_a, \mathbf{D}_\beta] = S(\gamma_a)_\beta{}^\gamma \mathbf{D}_\gamma , \quad [\mathbf{D}_a, \mathbf{D}_b] = -4S^2\mathcal{M}_{ab} , \quad (4.5b)$$

together with their complex conjugates. Upon a redefinition of the AdS parameter,  $S \rightarrow \rho/4$ , these (anti) commutation relations become identical to those which define the  $(2,0)$  AdS geometry introduced in [13].

### 4.2.2 (1,1) AdS superspace

Now, let us turn to the (1,1) case. The algebra (2.32a)–(2.32e) becomes

$$\{\mathcal{D}_\alpha^1, \mathcal{D}_\beta^1\} = 2i\mathcal{D}_{\alpha\beta} - 4iS\mathcal{M}_{\alpha\beta}, \quad \{\mathcal{D}_\alpha^2, \mathcal{D}_\beta^2\} = 2i\mathcal{D}_{\alpha\beta} + 4iS\mathcal{M}_{\alpha\beta}, \quad \{\mathcal{D}_\alpha^1, \mathcal{D}_\beta^2\} = 0, \quad (4.6a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^1] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^1, \quad [\mathcal{D}_a, \mathcal{D}_\beta^2] = -S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^2, \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}. \quad (4.6b)$$

We can introduce a complex basis for the covariant derivatives defined by

$$\nabla_\alpha = \frac{e^{i\varphi}}{\sqrt{2}}(\mathcal{D}_\alpha^1 - i\mathcal{D}_\alpha^2), \quad \bar{\nabla}_\alpha = -\frac{e^{-i\varphi}}{\sqrt{2}}(\mathcal{D}_\alpha^1 + i\mathcal{D}_\alpha^2), \quad (4.7)$$

with  $\varphi$  an arbitrary constant real phase. Then, the (anti) commutation relations (4.6a)–(4.6b) turn into

$$\{\nabla_\alpha, \nabla_\beta\} = -4\bar{\mu}\mathcal{M}_{\alpha\beta}, \quad \{\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} = 4\mu\mathcal{M}_{\alpha\beta}, \quad \{\nabla_\alpha, \bar{\nabla}_\beta\} = -2i\nabla_{\alpha\beta}, \quad (4.8a)$$

$$[\nabla_a, \nabla_\beta] = i\bar{\mu}(\gamma_a)_\beta{}^\gamma \bar{\nabla}_\gamma, \quad [\nabla_a, \bar{\nabla}_\beta] = -i\mu(\gamma_a)_\beta{}^\gamma \nabla_\gamma, \quad [\nabla_a, \nabla_b] = -4|\mu|^2\mathcal{M}_{ab}, \quad (4.8b)$$

where

$$\mu := -ie^{2i\varphi}S. \quad (4.9)$$

This is exactly the algebra of (1,1) AdS covariant derivatives [13].

## 4.3 $\mathcal{N} = 3$

There are two AdS superspaces in the  $\mathcal{N} = 3$  case: (3,0) and (2,1).

### 4.3.1 (3,0) AdS superspace

Let us start with the (3,0) AdS geometry described by

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4iS\delta^{IJ}\mathcal{M}_{\alpha\beta} - 4iS\varepsilon_{\alpha\beta}\mathcal{N}^{IJ}, \quad (4.10a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^J, \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}. \quad (4.10b)$$

As shown in [6], in order to define important off-shell supermultiplets and matter couplings in  $\mathcal{N} = 3$  conformal supergravity, it is useful to introduce a new basis for the spinor covariant derivatives,  $\mathcal{D}_\alpha^I \rightarrow \mathcal{D}_\alpha^{ij}$ , defined by the rule that any  $\text{SO}(3)$  isovector index is

replaced by a pair of symmetric SU(2) isospinor indices. Specifically, the new covariant derivative  $\mathcal{D}_\alpha^{ij}$  is defined as<sup>9</sup>

$$\mathcal{D}_\alpha^{ij} := \mathcal{D}_\alpha^I (\tau_I)^{ij} = \mathcal{D}_\alpha^{ji} , \quad (\mathcal{D}_\alpha^{ij})^* = -\mathcal{D}_{\alpha ij} = -\varepsilon_{ik} \varepsilon_{jl} \mathcal{D}_\alpha^{kl} , \quad i = \underline{1}, \underline{2} , \quad \varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1 . \quad (4.11)$$

In isospinor notations the SO(3) generator  $\mathcal{N}_{KL}$  becomes

$$\mathcal{N}_{KL} \rightarrow \mathcal{N}_{ijkl} := \mathcal{N}_{KL} (\tau^K)_{ij} (\tau^L)_{kl} = \frac{1}{2} \varepsilon_{jl} \mathcal{J}_{ik} + \frac{1}{2} \varepsilon_{ik} \mathcal{J}_{jl} , \quad (4.12a)$$

$$[\mathcal{J}^{kl}, \mathcal{D}_\alpha^{ij}] = \varepsilon^{i(k} \mathcal{D}_\alpha^{l)j} + \varepsilon^{j(k} \mathcal{D}_\alpha^{l)i} , \quad (4.12b)$$

where  $\mathcal{J}^{kl} = \mathcal{J}^{lk}$  is the SU(2) generator. In isospinor notations the (3,0) algebra takes the form

$$\{\mathcal{D}_\alpha^{ij}, \mathcal{D}_\beta^{kl}\} = -2i \varepsilon^{i(k} \varepsilon^{l)j} \mathcal{D}_{\alpha\beta} + 2i S \varepsilon_{\alpha\beta} \left( \varepsilon^{jl} \mathcal{J}^{ik} + \varepsilon^{ik} \mathcal{J}^{jl} \right) + 4i S \varepsilon^{i(k} \varepsilon^{l)j} \mathcal{M}_{\alpha\beta} , \quad (4.13a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{jk}] = S (\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{jk} , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 \mathcal{M}_{ab} . \quad (4.13b)$$

### 4.3.2 (2,1) AdS superspace

In the diagonal frame, the (2,1) algebra is

$$\{\mathcal{D}_\alpha^{\bar{I}}, \mathcal{D}_\beta^{\bar{J}}\} = 2i \delta^{\bar{I}\bar{J}} \mathcal{D}_{\alpha\beta} - 4i S \delta^{\bar{I}\bar{J}} \mathcal{M}_{\alpha\beta} - 4i S \varepsilon_{\alpha\beta} \mathcal{N}^{\bar{I}\bar{J}} , \quad (4.14a)$$

$$\{\mathcal{D}_\alpha^{\mathbf{3}}, \mathcal{D}_\beta^{\mathbf{3}}\} = 2i \mathcal{D}_{\alpha\beta} + 4i S \mathcal{M}_{\alpha\beta} , \quad \{\mathcal{D}_\alpha^{\bar{I}}, \mathcal{D}_\beta^{\mathbf{3}}\} = 0 , \quad (4.14b)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{\bar{I}}] = S (\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{\bar{I}} , \quad [\mathcal{D}_a, \mathcal{D}_\beta^{\mathbf{3}}] = -S (\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{\mathbf{3}} , \quad (4.14c)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 \mathcal{M}_{ab} . \quad (4.14d)$$

We want to rewrite the previous algebra in isospinor notations. To do that, we first observe that the algebra is constructed from the AdS algebra in the general frame (2.21a)–(2.21c) with the choice  $S^{IJ} = S (\delta^{IJ} - (w_3)^I (w_3)^J)$  with the vector  $(w_3)^I = (0, 0, \sqrt{2})$  in the third direction. It is clear that with an SO(3) rotation we can move to a general frame where  $S^{IJ} = S (\delta^{IJ} - w^I w^J)$  and  $w^I$  such that  $w^I w_I = 2$ . Clearly, the structure group is still SO(2) since, for example, the algebra admits a central extension with constant central charge field strength given by<sup>10</sup>  $w^{IJ} = \varepsilon^{IJK} w_K$ ,  $\mathcal{D}_A w_K = 0$ . The algebra (4.14a)–(4.14d) can be seen to become

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i \delta^{IJ} \mathcal{D}_{\alpha\beta} - 4i S (\delta^{IJ} - w^I w^J) \mathcal{M}_{\alpha\beta} - i S \varepsilon_{\alpha\beta} w^{IJ} w^{KL} \mathcal{N}_{KL} , \quad (4.15a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S (\delta_K^J - w^K w_J) (\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^K , \quad (4.15b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 \mathcal{M}_{ab} . \quad w^I w_I = 2 , \quad w^{IJ} := \varepsilon^{IJK} w_K . \quad (4.15c)$$

<sup>9</sup>We refer the reader to section 5 and Appendix A of [6] for details on our  $\mathcal{N} = 3$  isospinor notations including the properties and explicit definition of the  $(\tau^I)_{ij}$  matrices.

<sup>10</sup>See [6] for the description of  $\mathcal{N}$ -extended vector multiplets coupled to conformal supergravity. The same analysis holds for the AdS geometries.

From this form it is easy to move to isospinor notations in a general frame. We obtain

$$\{\mathcal{D}_\alpha^{ij}, \mathcal{D}_\beta^{kl}\} = -2i\varepsilon^{i(k}\varepsilon^{l)j}\mathcal{D}_{\alpha\beta} + 4iS(\varepsilon^{i(k}\varepsilon^{l)j} + w^{ij}w^{kl})\mathcal{M}_{\alpha\beta} \\ + iS\varepsilon_{\alpha\beta}(\varepsilon^{i(k}w^{l)j} + \varepsilon^{j(k}w^{l)i})w^{pq}\mathcal{J}_{pq}, \quad (4.16a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{ij}] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{ij} - S w^{ij}w_{kl}(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{kl}, \quad (4.16b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}, \quad w^{kl}w_{kl} = 2. \quad (4.16c)$$

Note that in the algebra the  $R$ -symmetry group is generated by the  $U(1)$  operator  $w^{pq}\mathcal{J}_{pq}$ .

## 4.4 $\mathcal{N} = 4$

In the  $\mathcal{N} = 4$  case we have three different AdS geometries: (4,0); (3,1); (2,2).

### 4.4.1 (4,0) AdS superspace

We start with the (4,0) case. This is particularly interesting being the first geometry where the covariantly constant  $X^{IJKL}$  curvature can be used to deform the AdS geometry. Since  $X^{IJKL} = X\varepsilon^{IJKL}$  for  $\mathcal{N} = 4$ , the (4,0) algebra is

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4iS\delta^{IJ}\mathcal{M}_{\alpha\beta} + i\varepsilon_{\alpha\beta}(X\varepsilon^{IJKL}\mathcal{N}_{KL} - 4S\mathcal{N}^{IJ}), \quad (4.17a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^J, \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}. \quad (4.17b)$$

Note that the scalar  $X$  is a free parameter that does not affect the curvature of the body of AdS. In particular, we can freely add it also to the  $\mathcal{N} = 4$  Minkowski superspace. Its role is to deform the  $SO(4)$  part of the structure group. To see in details how the  $X$  field affects the algebra we change notations for the  $SO(4)$  isovector indices and move to pairs of  $SU(2)_L \times SU(2)_R$  isospinor indices making use of the isomorphism  $SO(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ . We define new covariant derivatives  $\mathcal{D}_\alpha^{i\bar{i}}$  as<sup>11</sup>

$$\mathcal{D}_\alpha^{i\bar{i}} := \mathcal{D}_\alpha^I(\tau_I)^{i\bar{i}}, \quad (\mathcal{D}_\alpha^{i\bar{i}})^* = -\mathcal{D}_{\alpha i\bar{i}} = -\varepsilon_{ij}\varepsilon_{i\bar{j}}\mathcal{D}_\alpha^{j\bar{j}}. \quad (4.18)$$

The  $SO(4)$  generator  $\mathcal{N}_{KL}$  in isospinor notation takes the form

$$\mathcal{N}_{KL} \rightarrow \mathcal{N}_{k\bar{k}l\bar{l}} := \mathcal{N}_{KL}(\tau^K)_{k\bar{k}}(\tau^L)_{l\bar{l}} = \varepsilon_{\bar{k}l}\mathbf{L}_{kl} + \varepsilon_{kl}\mathbf{R}_{\bar{k}\bar{l}}, \quad (4.19a)$$

$$[\mathbf{L}^{kl}, \mathcal{D}_\alpha^{i\bar{i}}] = \varepsilon^{i(k}\mathcal{D}_\alpha^{l)\bar{i}}, \quad [\mathbf{R}^{\bar{k}\bar{l}}, \mathcal{D}_\alpha^{i\bar{i}}] = \varepsilon^{\bar{i}(\bar{k}}\mathcal{D}_\alpha^{i\bar{l})}, \quad (4.19b)$$

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<sup>11</sup>We refer the reader to section 6 and Appendix A of [6] for details on our  $\mathcal{N} = 4$  isospinor notations including the properties and explicit definition of the  $(\tau^I)_{i\bar{i}}$  matrices.

where  $\mathbf{L}_{kl}$  and  $\mathbf{R}_{\bar{k}\bar{l}}$  are respectively the left and right SU(2) generators. Finally, the (4,0) algebra becomes

$$\{\mathcal{D}_\alpha^{i\bar{i}}, \mathcal{D}_\beta^{j\bar{j}}\} = 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}(2S+X)\mathbf{L}^{ij} + 2i\varepsilon_{\alpha\beta}\varepsilon^{ij}(2S-X)\mathbf{R}^{\bar{i}\bar{j}} - 4iS\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{M}_{\alpha\beta} , \quad (4.20a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{j\bar{j}}] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{j\bar{j}} , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab} . \quad (4.20b)$$

It is interesting to note that for generic value of  $X$  the entire SO(4) group has non-trivial curvature in the algebra. But there are two points in which either the SU(2)<sub>R</sub> or the SU(2)<sub>L</sub> curvatures are zero and the structure group is reduced. These are given by

$$X = \pm 2S . \quad (4.21)$$

#### 4.4.2 (2,2) AdS superspace

The next case we consider is the (2,2) geometry. In the diagonal frame this takes exactly the form (2.32c)–(2.32e) where the  $\mathcal{N}^{\bar{I}J}$  rotates the directions  $I = \mathbf{1}, \mathbf{2}$  and the  $\mathcal{N}^{IJ}$  rotates the directions  $I = \mathbf{3}, \mathbf{4}$  in the isovector space. The torsion  $S^{IJ} = S \text{diag}(1, 1, -1, -1)$  is traceless  $\delta_{IJ}S^{IJ} = 0$ . We can use this information to derive the (2,2) geometry in isospinor notations. The traceless condition tells us that

$$S^{IJ} \rightarrow (\tau_I)^{\bar{i}i}(\tau_J)^{\bar{j}j}S^{IJ} = \mathcal{S}^{i\bar{j}\bar{i}j} = \mathcal{S}^{j\bar{i}i\bar{j}} = \mathcal{S}^{i\bar{j}\bar{i}j} , \quad (4.22)$$

which can be easily seen by remembering that [6]

$$\delta^{IJ} \rightarrow (\tau_I)^{\bar{i}i}(\tau_J)^{\bar{j}j}\delta^{IJ} = \varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} . \quad (4.23)$$

The constraint (2.20) in isospinor notation gives the condition

$$\mathcal{S}^{i\bar{j}\bar{i}j} = S l^{ij} r^{\bar{i}\bar{j}} , \quad l^{kl}l_{kl} = r^{\bar{k}\bar{l}}r_{\bar{k}\bar{l}} = 2 . \quad (4.24)$$

In a general frame, in isospinor notations, the (2,2) algebra then takes the following form

$$\{\mathcal{D}_\alpha^{i\bar{i}}, \mathcal{D}_\beta^{j\bar{j}}\} = 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} - 2iS\varepsilon_{\alpha\beta}\varepsilon^{ij}r^{\bar{i}\bar{j}}\mathbf{L} - 2iS\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}l^{ij}\mathbf{R} - 4iS l^{ij}r^{\bar{i}\bar{j}}\mathcal{M}_{\alpha\beta} , \quad (4.25a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{j\bar{j}}] = S l^j{}_k r^{\bar{j}}{}_{\bar{k}}(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{k\bar{k}} , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab} , \quad (4.25b)$$

where we have defined the U(1)<sub>L</sub> and U(1)<sub>R</sub> generators

$$\mathbf{L} := l^{kl}\mathbf{L}_{kl} , \quad \mathbf{R} := r^{\bar{k}\bar{l}}\mathbf{R}_{\bar{k}\bar{l}} . \quad (4.26)$$

### 4.4.3 (3,1) AdS superspace

We are left with the (3,1) case. In the diagonal frame the geometry is

$$\{\mathcal{D}_\alpha^{\bar{I}}, \mathcal{D}_\beta^{\bar{J}}\} = 2i\delta^{\bar{I}\bar{J}}\mathcal{D}_{\alpha\beta} - 4iS\delta^{\bar{I}\bar{J}}\mathcal{M}_{\alpha\beta} - 4iS\varepsilon_{\alpha\beta}\mathcal{N}^{\bar{I}\bar{J}}, \quad (4.27a)$$

$$\{\mathcal{D}_\alpha^{\bar{4}}, \mathcal{D}_\beta^{\bar{4}}\} = 2i\mathcal{D}_{\alpha\beta} + 4iS\mathcal{M}_{\alpha\beta}, \quad \{\mathcal{D}_\alpha^{\bar{I}}, \mathcal{D}_\beta^{\bar{4}}\} = \{\mathcal{D}_\alpha^{\bar{4}}, \mathcal{D}_\beta^{\bar{J}}\} = 0, \quad (4.27b)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{\bar{J}}] = S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{\bar{J}}, \quad [\mathcal{D}_a, \mathcal{D}_\beta^{\bar{4}}] = -S(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{\bar{4}}, \quad (4.27c)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}, \quad (4.27d)$$

where here  $\mathcal{N}^{\bar{I}\bar{J}}$  generate  $\text{SO}(3)$  rotations of the  $I = \mathbf{1}, \mathbf{2}, \mathbf{3}$  isovector indices leaving invariant the  $(w_4)^I = (0, 0, 0, \sqrt{2})$  vector. Similarly to the (2,1) case, with a  $\text{SO}(4)$  rotation we can rewrite the (3,1) geometry in a general frame as

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2i\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4iS(\delta^{IJ} - w^I w^J)\mathcal{M}_{\alpha\beta} - 4i\varepsilon_{\alpha\beta}S\hat{\mathcal{N}}^{IJ}, \quad (4.28a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^J] = S(\delta_K^J - w^K w_K)(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^K, \quad (4.28b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}, \quad \hat{\mathcal{N}}^{IJ} := (\delta^{K[I} - w^K w^{I]} \mathcal{N}_K^{J]}) , \quad (4.28c)$$

with  $w^I$  satisfying  $w^I w_I = 2$  but otherwise an arbitrary isovector. The operator  $\hat{\mathcal{N}}^{IJ}$  generates an  $\text{SO}(3)$  algebra inside  $\text{SO}(4)$ . This can be easily seen by observing that  $w^I$  is left invariant,  $\hat{\mathcal{N}}^{KL}w^I = 0$  and then  $\hat{\mathcal{N}}^{IJ}$  generates rotations orthogonal to  $w^I$ . Note that the previous representation of the (3,1) algebra is the same in describing the general  $(\mathcal{N} - 1, 1)$  cases. By using (4.28a)–(4.28c) we can derive a representation of the (3,1) algebra in isospinor notations. This takes the form

$$\begin{aligned} \{\mathcal{D}_\alpha^{\bar{i}\bar{i}}, \mathcal{D}_\beta^{\bar{j}\bar{j}}\} &= 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}S\left(\varepsilon^{\bar{i}\bar{j}}(\mathbf{L}^{ij} + w^i_{\bar{k}}w^j_{\bar{l}}\mathbf{R}^{\bar{k}\bar{l}}) + \varepsilon^{ij}(\mathbf{R}^{\bar{i}\bar{j}} + w_k^{\bar{i}}w_l^{\bar{j}}\mathbf{L}^{kl})\right) \\ &\quad - 4iS(\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} - w^{\bar{i}\bar{i}}w^{\bar{j}\bar{j}})\mathcal{M}_{\alpha\beta}, \end{aligned} \quad (4.29a)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{\bar{j}\bar{j}}] = S(\delta_k^j\delta_{\bar{k}}^{\bar{j}} - w^{\bar{j}\bar{j}}w_{k\bar{k}})(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{k\bar{k}}, \quad (4.29b)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4S^2\mathcal{M}_{ab}, \quad w^{k\bar{k}}w_{k\bar{k}} = 2, \quad w^{k\bar{k}}w_{j\bar{k}} = \delta_j^k, \quad w^{k\bar{k}}w_{k\bar{j}} = \delta_j^{\bar{k}}. \quad (4.29c)$$

Note that the spinor covariant derivatives algebra can be rewritten as

$$\begin{aligned} \{\mathcal{D}_\alpha^{\bar{i}\bar{i}}, \mathcal{D}_\beta^{\bar{j}\bar{j}}\} &= 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}S\left(\varepsilon^{\bar{i}\bar{j}}\delta_k^i\delta_l^j + \varepsilon^{ij}w_k^{\bar{i}}w_l^{\bar{j}}\right)\mathcal{J}^{kl} \\ &\quad - 4iS(\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} - w^{\bar{i}\bar{i}}w^{\bar{j}\bar{j}})\mathcal{M}_{\alpha\beta}, \end{aligned} \quad (4.30)$$

or equivalently as

$$\begin{aligned} \{\mathcal{D}_\alpha^{\bar{i}\bar{i}}, \mathcal{D}_\beta^{\bar{j}\bar{j}}\} &= 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}S\left(\varepsilon^{ij}\delta_k^{\bar{i}}\delta_l^{\bar{j}} + \varepsilon^{\bar{i}\bar{j}}w_k^i w_l^j\right)\mathcal{J}^{k\bar{l}} \\ &\quad - 4iS(\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} - w^{\bar{i}\bar{i}}w^{\bar{j}\bar{j}})\mathcal{M}_{\alpha\beta}, \end{aligned} \quad (4.31)$$

where we have defined

$$\mathcal{J}^{kl} := (\mathbf{L}^{kl} + w_{\bar{k}}^k w_{\bar{l}}^l \mathbf{R}^{\bar{k}\bar{l}}) , \quad (4.32a)$$

$$\mathcal{J}^{\bar{k}\bar{l}} := (\mathbf{R}^{\bar{k}\bar{l}} + w_k^{\bar{k}} w_l^{\bar{l}} \mathbf{L}^{kl}) . \quad (4.32b)$$

The operator  $\mathcal{J}^{kl} = w_{\bar{k}}^k w_{\bar{l}}^l \mathcal{J}^{\bar{k}\bar{l}}$ , or equivalently  $\mathcal{J}^{\bar{k}\bar{l}} = w_k^{\bar{k}} w_l^{\bar{l}} \mathcal{J}^{kl}$ , generates the residual SU(2) algebra of the (3,1) AdS geometry and leaves  $w^{\bar{i}\bar{i}}$  invariant.

## 5 Rigid $\mathcal{N} = 3$ supersymmetric field theories in AdS: Off-shell multiplets and invariant actions

In this and the next sections, our goal is to apply the supergravity techniques of [6] to describe general nonlinear  $\sigma$ -models in AdS<sub>3</sub> possessing  $\mathcal{N} = 3$  supersymmetry. We recall that the case of  $\mathcal{N} = 2$  AdS supersymmetry has already been studied in [13]. Similar in some aspects to  $\mathcal{N} = 3$ , the case of  $\mathcal{N} = 4$  AdS supersymmetry nevertheless requires a separate analysis that will be given elsewhere.

In discussing off-shell supermultiplets and supersymmetric actions, we first give a unified presentation that applies equally well to the (3,0) and (2,1) AdS supersymmetry types. After that, we spell out those technical aspects of  $\mathcal{N} = 3$  supersymmetric theories in AdS<sub>3</sub> which look essentially different for the cases (3,0) and (2,1).

For our subsequent consideration, it is useful to rewrite the (anti) commutation relations for the (3,0) and (2,1) covariant derivatives in a unified form (which is inspired by the algebra of covariant derivatives in  $\mathcal{N} = 3$  conformal supergravity [13]):

$$\begin{aligned} \{\mathcal{D}_{\alpha}^{ij}, \mathcal{D}_{\beta}^{kl}\} = & -2i\varepsilon^{i(k}\varepsilon^{l)j}\mathcal{D}_{\alpha\beta} - 4i(\mathcal{S}^{ijkl} - \varepsilon^{i(k}\varepsilon^{l)j}\mathcal{S})\mathcal{M}_{\alpha\beta} \\ & -i\varepsilon_{\alpha\beta}(\varepsilon^{jl}\mathcal{S}^{ikpq} + \varepsilon^{ik}\mathcal{S}^{jlpq})\mathcal{J}_{pq} + 2i\varepsilon_{\alpha\beta}\mathcal{S}\left(\varepsilon^{jl}\mathcal{J}^{ik} + \varepsilon^{ik}\mathcal{J}^{jl}\right) , \end{aligned} \quad (5.1a)$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma}^{ij}] = -2\mathcal{S}^{ijkl}\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)kl} - 2\mathcal{S}\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}^{ij} . \quad (5.1b)$$

In (5.1a)–(5.5c), the covariantly constant tensors  $\mathcal{S}^{ijkl} = \mathcal{S}^{(ijkl)}$  and  $\mathcal{S}$  have the following explicit expressions for the (3,0) and (2,1) AdS superspaces

$$(3,0) \text{ AdS : } \quad \mathcal{S} = S , \quad \mathcal{S}^{ijkl} = 0 ; \quad (5.2)$$

$$(2,1) \text{ AdS : } \quad \mathcal{S} = \frac{1}{3}S , \quad \mathcal{S}^{ijkl} = -S w^{(ij} w^{kl)} , \quad (5.3)$$

where the covariantly constant tensor  $w^{ij} = w^{(ij)}$  is normalised by  $w^{ij}w_{ij} = 2$ .

The  $\mathcal{N} = 3$  Killing equations

$$\left[\xi + \frac{1}{2}\Lambda^{ab}\mathcal{M}_{ab} + \frac{1}{2}\Lambda^{ij}\mathcal{J}_{ij}, \mathcal{D}_C\right] = 0, \quad \xi = \xi^a\mathcal{D}_a + \xi_{ij}^\alpha\mathcal{D}_\alpha^{ij} \quad (5.4)$$

are equivalent to

$$\mathcal{D}_\alpha^{ij}\xi_\beta^{kl} = \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}\Lambda^{jl} + \varepsilon^{jl}\Lambda^{ik}) + (\mathcal{S}^{ijkl} - \varepsilon^{i(k}\varepsilon^{l)j}\mathcal{S})\xi_{\alpha\beta} - \frac{1}{2}\varepsilon^{i(k}\varepsilon^{l)j}\Lambda_{\alpha\beta}, \quad (5.5a)$$

$$0 = \mathcal{D}_\gamma^{ij}\xi^{\alpha\gamma} + 6i\xi^{\alpha ij}, \quad 0 = \mathcal{D}_\gamma^{ij}\Lambda^{\alpha\gamma} + 12i\xi_{kl}^\alpha(\mathcal{S}^{ijkl} - \varepsilon^{i(k}\varepsilon^{l)j}\mathcal{S}) \quad (5.5b)$$

$$0 = \mathcal{D}_{(\alpha}^{ij}\xi_{\beta\gamma)} = \mathcal{D}_{(\alpha}^{ij}\Lambda_{\beta\gamma)}. \quad (5.5c)$$

These relations imply, in particular, the following equations

$$\mathcal{D}_{\alpha kl}\Lambda^{kl} = 0, \quad \mathcal{D}_\alpha^{(i}{}_{k}\Lambda^{j)k} = -2i\left(4\mathcal{S}\xi_\alpha^{ij} + \mathcal{S}^{ijkl}\xi_{\alpha kl}\right), \quad \mathcal{D}_\alpha^{(ij}\Lambda^{kl)} = -4i\xi_\alpha^{(i}{}_{p}\mathcal{S}^{jkl)p} \quad (5.6)$$

which will be important for our subsequent consideration. We recall that the parameter  $\Lambda^{ij}$  is real,  $\overline{\Lambda^{ij}} = \Lambda_{ij}$ .

In the (3,0) and (2,1) cases, the  $R$ -symmetry groups are  $SU(2)$  and  $U(1)$  respectively. In the case of (2,1) AdS supersymmetry, the parameter  $\Lambda^{ij}$  has the form

$$(2,1) \text{ AdS} : \quad \Lambda^{ij} = w^{ij}\Lambda, \quad \Lambda = \overline{\Lambda}. \quad (5.7)$$

## 5.1 Covariant projective supermultiplets

In complete analogy with matter couplings in  $\mathcal{N} = 3$  supergravity [6], a large class of rigid supersymmetric theories in (3,0) and (2,1) AdS superspaces can be formulated in terms of covariant projective supermultiplets. Before introducing these supermultiplet, a few words are in order regarding the so-called projective superspace approach.

The projective superspace approach [30, 31, 32] is a method to construct off-shell 4D  $\mathcal{N} = 2$  super-Poincaré invariant theories in the superspace  $\mathbb{R}^{4|8} \times \mathbb{CP}^1$  introduced for the first time by Rosly [33].<sup>12</sup> The most important projective supermultiplets are: the  $\mathcal{O}(1)$  multiplet [33] (equivalent to the on-shell hypermultiplet [37]); the real  $\mathcal{O}(2)$  multiplet [30] (equivalent to the  $\mathcal{N} = 2$  tensor multiplet [38]); the  $\mathcal{O}(n)$  multiplets [39, 31], where  $n = 3, 4, \dots$ ; the polar (arctic + antarctic) multiplet [31]; the tropical multiplet [32]. These multiplets are off-shell except the  $\mathcal{O}(1)$  multiplet. The projective superspace approach

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<sup>12</sup>The same superspace is used within the harmonic superspace approach [34, 35] which is more general than the projective one but less useful for various  $\sigma$ -model applications. The precise relationship between the harmonic and projective superspace formulations is spelled out in [36].



was extended to conformal supersymmetry [40, 41] and supergravity [42, 43], more than twenty years after the original publication on self-interacting  $\mathcal{N} = 2$  tensor multiplets [30]. The original 5D  $\mathcal{N} = 1$  supergravity construction of [42, 43] has successfully been extended to 4D  $\mathcal{N} = 2$  supergravity [44, 45], 3D  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  supergravity theories [13], 2D  $\mathcal{N} = (4, 4)$  supergravity [46], and most recently 6D  $\mathcal{N} = (1, 0)$  supergravity [47].

A *covariant projective supermultiplet* of weight  $n$ ,  $Q^{(n)}(z^M, v^i)$ , is defined to be a Lorentz-scalar superfield that lives on the appropriate  $\mathcal{N} = 3$  AdS superspace  $\mathcal{M}^{3|6}$  (which is  $\text{AdS}_{(3|3,0)}$  or  $\text{AdS}_{(3|2,1)}$ ), is holomorphic with respect to isospinor variables  $v^i$  on an open domain of  $\mathbb{C}^2 \setminus \{0\}$ , and is characterised by the following conditions:

(i)  $Q^{(n)}$  is a homogeneous function of  $v$  of degree  $n$ , that is,

$$Q^{(n)}(z, cv) = c^n Q^{(n)}(z, v), \quad c \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}; \quad (5.8)$$

(ii) Under the appropriate AdS isometry supergroup, which is  $\text{OSp}(3|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$  or  $\text{OSp}(2|2; \mathbb{R}) \times \text{OSp}(1|2; \mathbb{R})$ ,  $Q^{(n)}$  transforms as follows:

$$\begin{aligned} \delta_\xi Q^{(n)} &= \left( \xi + \frac{1}{2} \Lambda^{ij} \mathcal{J}_{ij} \right) Q^{(n)}, \\ \Lambda^{ij} \mathcal{J}_{ij} Q^{(n)} &= - \left( \Lambda^{(2)} \boldsymbol{\partial}^{(-2)} - n \Lambda^{(0)} \right) Q^{(n)}, \quad \boldsymbol{\partial}^{(-2)} := \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i}. \end{aligned} \quad (5.9)$$

where  $\xi$  denotes an arbitrary AdS Killing vector field, eq. (2.33), and  $\Lambda^{ij}$  the associated  $\text{SU}(2)$  parameter defined by (2.34).<sup>13</sup> In eq. (5.9), we have introduced

$$\Lambda^{(2)} := \Lambda^{ij} v_i v_j, \quad \Lambda^{(0)} := \frac{v_i u_j}{(v, u)} \Lambda^{ij}, \quad (v, u) := v^i u_i. \quad (5.10)$$

The transformation law (5.9) involves an additional isotwistor  $u_i$ , which is only subject to the condition  $(v, u) \neq 0$ , and otherwise is completely arbitrary. Both  $Q^{(n)}$  and  $\delta_\xi Q^{(n)}$  are independent of  $u_i$ .

(iii)  $Q^{(n)}$  obeys the analyticity constraint

$$\mathcal{D}_\alpha^{(2)} Q^{(n)} = 0, \quad \mathcal{D}_\alpha^{(2)} := v_i v_j \mathcal{D}_\alpha^{ij}. \quad (5.11)$$

The analyticity constraint (5.11) and the homogeneity condition (5.8) are consistent with the interpretation that the isospinor  $v^i \in \mathbb{C}^2 \setminus \{0\}$  is defined modulo the equivalence

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<sup>13</sup>In the case of (2,1) AdS supersymmetry, the parameter  $\Lambda^{ij}$  is constrained to be  $\Lambda^{ij} = \Lambda w^{ij}$ , which corresponds to an  $\text{SO}(2)$  subgroup of  $\text{SU}(2)$ .

relation  $v^i \sim c v^i$ , with  $c \in \mathbb{C}^*$ , hence it parametrizes  $\mathbb{CP}^1$ . Therefore, the projective multiplets live in  $\mathcal{M}^{3|6} \times \mathbb{CP}^1$ .

Two comments are in order. Firstly, it follows from eq. (5.9) that

$$\mathcal{J}^{(2)} Q^{(n)} = 0 \ , \quad \mathcal{J}^{(2)} := v_i v_j \mathcal{J}^{ij} \ . \quad (5.12)$$

Secondly, the constraints (5.11) are fully consistent due to the facts that  $Q^{(n)}$  is a Lorentz scalar, and the operators  $\mathcal{D}_\alpha^{(2)}$  obey the anti-commutation relations

$$\{\mathcal{D}_\alpha^{(2)}, \mathcal{D}_\beta^{(2)}\} = -4i\mathcal{S}^{(4)} \mathcal{M}_{\alpha\beta} \ , \quad (5.13)$$

with

$$\mathcal{S}^{(4)} := v_i v_j v_k v_l \mathcal{S}^{ijkl} \ . \quad (5.14)$$

A more general family of off-shell supermultiplets is obtained by removing the condition (iii) in the above definition, while keeping intact the conditions (i) and (ii). Such supermultiplets are called *isotwistor*. These superfields can be used to construct projective ones with the aid of the so-called analytic projection operator

$$\Delta^{(4)} := \frac{i}{4} \left( \mathcal{D}^{(4)} - 4i\mathcal{S}^{(4)} \right) \ , \quad \mathcal{D}^{(4)} := \mathcal{D}^{(2)\alpha} \mathcal{D}_\alpha^{(2)} \ . \quad (5.15)$$

If  $U^{(n-4)}(z, v)$  is an isotwistor superfield, then  $Q^{(n)}(z, v) := \Delta^{(4)} U^{(n-4)}(z, v)$  is a covariant projective superfield,

$$\mathcal{D}_\alpha^{(2)} \Delta^{(4)} U^{(n-4)} = 0 \ . \quad (5.16)$$

There exists a real structure on the space of projective multiplets known as the smile conjugation.<sup>14</sup> Given a weight- $n$  projective multiplet  $Q^{(n)}(v^i)$ , its *smile conjugate*,  $\check{Q}^{(n)}(v^i)$ , is defined by

$$Q^{(n)}(v^i) \longrightarrow \bar{Q}^{(n)}(\bar{v}_i) \longrightarrow \bar{Q}^{(n)}(\bar{v}_i \rightarrow -v_i) =: \check{Q}^{(n)}(v^i) \ , \quad (5.17)$$

with  $\bar{Q}^{(n)}(\bar{v}_i) := \overline{Q^{(n)}(v^i)}$  the complex conjugate of  $Q^{(n)}(v^i)$ , and  $\bar{v}_i$  the complex conjugate of  $v^i$ . One can show that  $\check{Q}^{(n)}(v)$  is a weight- $n$  projective multiplet. In particular,  $\check{Q}^{(n)}(v)$  obeys the analyticity constraint  $\mathcal{D}_\alpha^{(2)} \check{Q}^{(n)} = 0$ , unlike the complex conjugate of  $Q^{(n)}(v)$ . One can also check that

$$\check{\check{Q}}^{(n)}(v) = (-1)^n Q^{(n)}(v) \ . \quad (5.18)$$

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<sup>14</sup>The smile conjugation was pioneered by Rosly [33] and re-discovered in [34, 30].

Therefore, if  $n$  is even, one can define real projective multiplets,  $\check{Q}^{(2n)} = Q^{(2n)}$ . Note that geometrically, the smile-conjugation is complex conjugation composed with the antipodal map on the projective space  $\mathbb{CP}^1$ .

We now list several projective multiplets that can be used to describe superfield dynamical variables. A complex  $\mathcal{O}(m)$  multiplet, with  $m = 1, 2, \dots$ , is described by a weight- $m$  projective superfield  $H^{(m)}(v)$  of the form:

$$H^{(m)}(v) = H^{i_1 \dots i_m} v_{i_1} \dots v_{i_m} . \quad (5.19)$$

The analyticity constraint (5.11) is equivalent to

$$\mathcal{D}_\alpha^{(ij} H^{k_1 \dots k_m)} = 0 . \quad (5.20)$$

If  $m$  is even,  $m = 2n$ , we can define a real  $\mathcal{O}(2n)$  multiplet<sup>15</sup> obeying the reality condition  $\check{H}^{(2n)} = H^{(2n)}$ , or equivalently

$$\overline{H^{i_1 \dots i_{2n}}} = H_{i_1 \dots i_{2n}} = \varepsilon_{i_1 j_1} \dots \varepsilon_{i_{2n} j_{2n}} H^{j_1 \dots j_{2n}} . \quad (5.21)$$

The field strength of an Abelian vector multiplet is a real  $\mathcal{O}(2)$  multiplet [6]. For  $n > 1$ , the real  $\mathcal{O}(2n)$  multiplet can be used to describe an off-shell (neutral) hypermultiplet.

An off-shell (charged) hypermultiplet can be described in term of the so-called *arctic* weight- $n$  multiplet  $\Upsilon^{(n)}(v)$  which is defined to be holomorphic in the north chart  $\mathbb{C}$ , of the projective space  $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ :

$$\Upsilon^{(n)}(v) = (v^\perp)^n \Upsilon^{[n]}(\zeta) , \quad \Upsilon^{[n]}(\zeta) = \sum_{k=0}^{\infty} \Upsilon_k \zeta^k , \quad (5.22)$$

and its smile-conjugate *antarctic* multiplet  $\check{\Upsilon}^{(n)}(v)$ ,

$$\check{\Upsilon}^{(n)}(v) = (v^\perp)^n \check{\Upsilon}^{[n]}(\zeta) = (v^\perp)^n \check{\Upsilon}^{[n]}(\zeta) , \quad \check{\Upsilon}^{[n]}(\zeta) = \sum_{k=0}^{\infty} \bar{\Upsilon}_k \frac{(-1)^k}{\zeta^k} . \quad (5.23)$$

Here we have introduced the inhomogeneous complex coordinate  $\zeta = v^2/v^\perp$  on the north chart of  $\mathbb{CP}^1$ . The pair consisting of  $\Upsilon^{[n]}(\zeta)$  and  $\check{\Upsilon}^{[n]}(\zeta)$  constitutes the so-called polar weight- $n$  multiplet.

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<sup>15</sup>In 4D  $\mathcal{N} = 2$  Poincaré supersymmetry, the real  $\mathcal{O}(2n)$  multiplets, with  $n > 1$ , and their self-interactions were introduced for the first time by Ketov and Tyutin [39] and re-discovered in [32].

## 5.2 Supersymmetric action

In order to formulate the dynamics of rigid  $\mathcal{N} = 3$  supersymmetric field theories in  $\text{AdS}_3$ , a manifestly supersymmetric action principle is required. It can be readily constructed by restricting the locally supersymmetric action introduced in [6] to the appropriate AdS superspace. The action is generated by a Lagrangian  $\mathcal{L}^{(2)}(z, v)$ , which is a covariant weight-2 real projective multiplet, and has the form:

$$S[\mathcal{L}^{(2)}] = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \int d^3x d^6\theta E \mathcal{C}^{(-4)} \mathcal{L}^{(2)} , \quad E^{-1} = \text{Ber}(E_A^M) . \quad (5.24)$$

Here the line integral is carried out over a closed contour  $\gamma = \{v^i(t)\}$  in  $\mathbb{CP}^1$ . The action involves an isotwistor superfield  $\mathcal{C}^{(-4)}(z, v)$  defined by

$$\mathcal{C}^{(-4)} := \frac{\mathcal{U}^{(n)}}{\Delta^{(4)}\mathcal{U}^{(n)}} , \quad (5.25)$$

for some isotwistor multiplet  $\mathcal{U}^{(n)}$  such that  $1/\Delta^{(4)}\mathcal{U}^{(n)}$  is well defined. The superfield  $\mathcal{C}^{(-4)}$  is required to write the action as an integral over the full AdS superspace. It is actually a purely gauge degree of freedom in the sense that (5.24) is independent of the explicit choice of  $\mathcal{U}^{(n)}$ . Indeed, varying  $\mathcal{U}^{(n)}$  gives

$$\delta\mathcal{C}^{(-4)} = \frac{\delta\mathcal{U}^{(n)}}{\Delta^{(4)}\mathcal{U}^{(n)}} - \frac{\mathcal{U}^{(n)}\Delta^{(4)}\delta\mathcal{U}^{(n)}}{(\Delta^{(4)}\mathcal{U}^{(n)})^2} .$$

In the contribution to  $\delta S[\mathcal{L}^{(2)}]$  which comes from the second term, we can integrate by parts, to strip  $\delta\mathcal{U}^{(n)}$  of  $\Delta^{(4)}$ , and make use of the fact that  $\mathcal{L}^{(2)}$  and  $\Delta^{(4)}\mathcal{U}^{(n)}$  are covariant projective multiplets. As a result, we obtain  $\delta S[\mathcal{L}^{(2)}] = 0$ .

In the case of (2,1) AdS supersymmetry, there is a simple choice for  $\mathcal{C}^{(-4)}$ :

$$\mathcal{C}^{(-4)} = \frac{1}{\Delta^{(4)}1} = -\frac{1}{S(w^{(2)})^2} , \quad w^{(2)} := v_i v_j w^{ij} . \quad (5.26)$$

## 5.3 Supersymmetric action: Integrating out all the fermionic directions

The action (5.24) is manifestly invariant under arbitrary isometry transformations of the appropriate AdS superspace,  $\text{AdS}_{(3|3,0)}$  or  $\text{AdS}_{(3|2,1)}$ . The price to pay for this is two-fold: (i) the action involves the superfield  $\mathcal{C}^{(-4)}$  which is a purely gauge degree of freedom; (ii) the action is given by an integral over six Grassmann variables while the Lagrangian

$\mathcal{L}^{(2)}$  depends only on four of these coordinates. Both drawbacks can be eliminated, at the cost of losing the manifest invariance under the AdS isometry supergroup, if one integrates out *two* or *all* of the six fermionic directions. To achieve this, one could use the powerful method of normal coordinates around a submanifold of curved superspace [48]. Here we are going to use an alternative technique which was first developed to derive the  $\mathcal{N} = 1$  supersymmetric action in AdS<sub>5</sub> [49].

Our point of departure is the  $\mathcal{N} = 3$  projective superspace action in three-dimensional Minkowski space which was introduced in [12]. It has the form

$$S[L^{(2)}] = \frac{1}{8\pi} \oint_{\gamma} v_i dv^i \int d^3x (D^{(-2)})^2 (D^{(0)})^2 L^{(2)} \Big|_{\theta=0} , \quad (5.27)$$

where the Lagrangian  $L^{(2)}(z, v)$  is a real weight-two projective multiplet, and the operators  $D_{\alpha}^{(-2)}$  and  $D_{\alpha}^{(0)}$  are defined in terms of the *flat* spinor covariant derivatives  $D_{\alpha}^{ij}$  as follows

$$D_{\alpha}^{(-2)} := \frac{u_i u_j}{(v, u)^2} D_{\alpha}^{ij} , \quad D_{\alpha}^{(0)} := \frac{v_i u_j}{(v, u)} D_{\alpha}^{ij} . \quad (5.28)$$

These operators depend not only on the isotwistor  $v^i(t)$ , which varies along the integration contour, but also on a constant ( $t$ -independent) isotwistor  $u_i$  chosen in such a way that  $v_i(t)$  and  $u_i$  are linearly independent at each point of the contour  $\gamma$ , that is  $(v(t), u) \neq 0$ . The action (5.27) is actually independent of  $u_i$ , since it proves to be invariant under arbitrary projective transformations of the form

$$(u_i, v_i(t)) \rightarrow (u_i, v_i(t)) R(t) , \quad R(t) = \begin{pmatrix} a(t) & 0 \\ b(t) & c(t) \end{pmatrix} \in \text{GL}(2, \mathbb{C}) , \quad (5.29)$$

where the matrix elements  $a(t)$  and  $b(t)$  obey the first-order differential equations

$$\dot{a} = b \frac{(\dot{v}, v)}{(v, u)} , \quad \dot{b} = -b \frac{(\dot{v}, u)}{(v, u)} , \quad (5.30)$$

with  $\dot{f} := df(t)/dt$  for any function  $f(t)$ . This invariance follows from the following properties of the Lagrangian: (i)  $L^{(2)}(v)$  is a homogeneous function of  $v^i$  of degree two; and (ii)  $L^{(2)}(v)$  obeys the analyticity condition

$$D_{\alpha}^{(2)} L^{(2)}(v) = 0 , \quad D_{\alpha}^{(2)} := v_i v_j D_{\alpha}^{ij} . \quad (5.31)$$

It turns out that the property (ii) suffices to prove that the action (5.27) is invariant under the standard  $\mathcal{N} = 3$  super-Poincaré transformations in three dimensions [12].

We now try to generalise the above construction to the AdS case. Let  $z^M = (x^m, \theta_{ij}^\mu)$  be local coordinates of the AdS superspace. Given a tensor superfield  $U(x, \theta)$ , we define its restriction to the body of the superspace,  $\theta_{ij}^\mu = 0$ , specifically<sup>16</sup>

$$U|| := U(x, \theta)|_{\theta_{ij}^\mu=0} . \quad (5.32)$$

We also define the double-bar projection of the covariant derivatives

$$\mathcal{D}_A|| := E_A^M || \partial_M + \frac{1}{2} \Omega_A^{bc} || \mathcal{M}_{bc} + \frac{1}{2} \Phi_A^{kl} || \mathcal{J}_{kl} . \quad (5.33)$$

Since for both the (3,0) and (2,1) AdS geometries it holds that  $[\mathcal{D}_a, \mathcal{D}_b] = -4S^2 \mathcal{M}_{ab}$ , we can use the freedom to perform general coordinate and local structure group transformations to choose a (Wess-Zumino) gauge in which

$$\mathcal{D}_a|| = \nabla_a = e_a^m(x) \partial_m + \frac{1}{2} \omega_a^{bc}(x) \mathcal{M}_{bc} , \quad (5.34)$$

where  $\nabla_a$  stands for the covariant derivative of anti-de Sitter space  $\text{AdS}_3$ ,

$$[\nabla_a, \nabla_b] = -4S^2 \mathcal{M}_{ab} . \quad (5.35)$$

We are interested in constructing an AdS generalisation of the action (5.27). On general grounds, it should have the form

$$S[\mathcal{L}^{(2)}] = S_0 + \dots , \quad S_0 = \frac{1}{8\pi} \oint_\gamma v_i dv^i \int d^3x e (\mathcal{D}^{(-2)})^2 (\mathcal{D}^{(0)})^2 \mathcal{L}^{(2)}|| , \quad (5.36)$$

with  $e := \det^{-1}(e_m^a)$ . Note that in (5.36) the dots stand for curvature dependent corrections which are necessary for the action to be invariant under the symmetries of its parent action (5.24). It is interesting to note that there is one symmetry which is shared by the flat action (5.27) and the parent curved full superspace action (5.24): both are manifestly projective invariant (5.29). On the other hand  $S_0$  is not projective invariant. As discussed in [49, 42, 48], one can actually exploit projective invariance as a tool to iteratively find the completion of  $S_0$  to  $S[\mathcal{L}^{(2)}]$  in (5.36). In Appendix A we sketch how to describe this approach for the (3,0) and (2,1) AdS cases. Let us now write down the form of the full  $\mathcal{N} = 3$  AdS projective action principle in components

$$\begin{aligned} S[\mathcal{L}^{(2)}] = \frac{1}{8\pi} \oint_\gamma v_i dv^i \int d^3x e \Big[ & (\mathcal{D}^{(-2)})^2 (\mathcal{D}^{(0)})^2 + 4i(\mathcal{S} - 2\mathcal{S}^{(0)})(\mathcal{D}^{(-2)})^2 \\ & + 12i\mathcal{S}^{(-2)}\mathcal{D}^{(-2)\alpha}\mathcal{D}_\alpha^{(0)} - 16i\mathcal{S}^{(-4)}(\mathcal{D}^{(0)})^2 \\ & - 144\mathcal{S}^{(-2)}\mathcal{S}^{(-2)} + 64\mathcal{S}^{(-4)}\mathcal{S}^{(0)} + 48\mathcal{S}^{(-4)}\mathcal{S} \Big] \mathcal{L}^{(2)}|| . \end{aligned} \quad (5.37)$$

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<sup>16</sup>In what follows, we will also introduce a single bar-projection,  $U|$ , to be the restriction of  $U$  to a certain  $\mathcal{N} = 2$  subspace of the  $\mathcal{N} = 3$  AdS superspace under consideration.

Here we have used the definitions

$$\mathcal{S}^{(0)} := \frac{v_i v_j u_k u_l \mathcal{S}^{ijkl}}{(v, u)^2} , \quad \mathcal{S}^{(-2)} := \frac{v_i u_j u_k u_l \mathcal{S}^{ijkl}}{(v, u)^3} , \quad \mathcal{S}^{(-4)} := \frac{u_i u_j u_k u_l \mathcal{S}^{ijkl}}{(v, u)^4} . \quad (5.38)$$

The actions corresponding to the (3,0) and (2,1) AdS superspaces are obtained from (5.37) by choosing the curvature as follows:

$$(3,0) : \quad \mathcal{S} = S , \quad \mathcal{S}^{ijkl} = 0 , \quad (5.39a)$$

$$(2,1) : \quad \mathcal{S} = \frac{1}{3} S , \quad \mathcal{S}^{ijkl} = -S w^{(ij} w^{kl)} , \quad w^{ij} w_{ij} = 2 . \quad (5.39b)$$

## 6 Supersymmetric action: Reduction to $\mathcal{N} = 2$ superspace

The representation (5.37) obtained in the previous subsection, corresponds to the situation when all the Grassmann integrals in the action (5.24) have been done. Here we take a different course and reduce the superspace integral in (5.24) to that over a certain  $\mathcal{N} = 2$  subspace of the full  $\mathcal{N} = 3$  AdS superspace under consideration. Such a procedure cannot be carried out in a unified way for the cases (3,0) and (2,1), and thus a separate consideration should be given in each case.

### 6.1 AdS superspace reduction: (3,0) to (2,0)

To identify an  $\mathcal{N} = 2$  subspace of the  $\mathcal{N} = 3$  AdS superspace, we need a subset of four spinor covariant derivatives which, together with  $\mathcal{D}_a$ , lead to a closed set of (anti) commutation relations.

In the case of (3,0) AdS superspace, the covariant derivatives obey the (anti) commutation relations (4.13). A closed subalgebra can be identified with the mutually conjugate derivatives  $\mathcal{D}_\alpha^{11}$  and  $-\mathcal{D}_\alpha^{22}$  (for any bosonic superfield  $U$ , it holds that  $\overline{\mathcal{D}_\alpha^{11} U} = -\mathcal{D}_\alpha^{22} \overline{U}$ ). Indeed, it follows from (4.13) that

$$\{\mathcal{D}_\alpha^{11}, \mathcal{D}_\beta^{11}\} = \{(-\mathcal{D}_\alpha^{22}), (-\mathcal{D}_\beta^{22})\} = 0 , \quad (6.1a)$$

$$\{\mathcal{D}_\alpha^{11}, (-\mathcal{D}_\beta^{22})\} = -2i\mathcal{D}_{\alpha\beta} - 4i\varepsilon_{\alpha\beta} S \mathcal{J}^{12} + 4i S \mathcal{M}_{\alpha\beta} , \quad (6.1b)$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^{11}] = S (\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma^{11} , \quad [\mathcal{D}_a, (-\mathcal{D}_\beta^{22})] = S (\gamma_a)_\beta{}^\gamma (-\mathcal{D}_\gamma^{22}) , \quad (6.1c)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 \mathcal{M}_{ab} . \quad (6.1d)$$

For the subset  $(\mathcal{D}_a, \mathcal{D}_\alpha^{11}, -\mathcal{D}_\alpha^{22})$  chosen, the original  $R$ -symmetry group  $SU(2)$  reduces to  $U(1)$ , and the corresponding generator  $\mathcal{J}^{12}$  acts on the spinor derivatives as

$$[\mathcal{J}^{12}, \mathcal{D}_\alpha^{11}] = \mathcal{D}_\alpha^{11} , \quad [\mathcal{J}^{12}, (-\mathcal{D}_\alpha^{22})] = -(-\mathcal{D}_\alpha^{22}) . \quad (6.2)$$

The (anti) commutation relations (6.1) can be recognised as those corresponding to the  $(2,0)$  AdS superspace,  $\text{AdS}_{(3|2,0)}$ , studied in [13].

Now, we can embed the superspace  $\text{AdS}_{(3|2,0)}$  into  $\text{AdS}_{(3|3,0)}$ . Given a tensor superfield  $U(x, \theta_{ij})$  in  $\text{AdS}_{(3|3,0)}$ , we define its projection

$$U| := U(x, \theta_{ij})|_{\theta_{12}=0} . \quad (6.3)$$

By definition,  $U|$  still depends on the Grassmann coordinates  $\theta^\mu := \theta_{11}^\mu$  and their complex conjugate  $\bar{\theta}^\mu = \theta_{22}^\mu$ . For the  $(3,0)$  AdS covariant derivatives

$$\mathcal{D}_A = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} \mathcal{M}_{bc} + \frac{1}{2} \Phi_A^{kl} \mathcal{J}_{kl} , \quad (6.4)$$

the projection is defined as

$$\mathcal{D}_A| = E_A^M | \partial_M + \frac{1}{2} \Omega_A^{bc} | \mathcal{M}_{bc} + \frac{1}{2} \Phi_A^{kl} | \mathcal{J}_{kl} . \quad (6.5)$$

Since the operators  $(\mathcal{D}_a, \mathcal{D}_\alpha^{11}, -\mathcal{D}_\alpha^{22})$  form a closed algebra, which is isomorphic to that of the covariant derivatives for  $\text{AdS}_{(3|2,0)}$ , one can use the freedom to perform general coordinate, local Lorentz and  $SU(2)$  transformations to chose a gauge in which

$$\mathcal{D}_\alpha^{11}| = \mathbf{D}_\alpha , \quad (-\mathcal{D}_\alpha^{22})| = \bar{\mathbf{D}}_\alpha , \quad (6.6)$$

where

$$\mathbf{D}_A = (\mathbf{D}_a, \mathbf{D}_\alpha, \bar{\mathbf{D}}^\alpha) = \mathbf{E}_A^M \partial_M + \frac{1}{2} \mathbf{\Omega}_A^{cd} \mathcal{M}_{cd} + \mathbf{i} \mathbf{\Phi}_A \mathcal{J} \quad (6.7)$$

denote the covariant derivatives of  $\text{AdS}_{(3|2,0)}$  which obey the (anti) commutation relations (4.5), with  $\mathcal{J} \equiv \mathcal{J}^{12}$ . In such a coordinate system<sup>17</sup> the operators  $\mathcal{D}_\alpha^{11}|$  and  $\mathcal{D}_\alpha^{22}|$  involve no partial derivative with respect to  $\theta_{12}$ , and therefore, for any positive integer  $k$ , it holds that  $(\mathcal{D}_{\hat{\alpha}_1} \cdots \mathcal{D}_{\hat{\alpha}_k} U)| = \mathcal{D}_{\hat{\alpha}_1}| \cdots \mathcal{D}_{\hat{\alpha}_k}| U|$ , where  $\mathcal{D}_{\hat{\alpha}} := (\mathcal{D}_\alpha^{11}, -\mathcal{D}_\alpha^{22})$  and  $U$  is a tensor superfield. This implies that  $\mathcal{D}_a| = \mathbf{D}_a$ .

Our next task is to reduce the transformation laws of projective supermultiplets from  $\text{AdS}_{(3|3,0)}$  to its  $\mathcal{N} = 2$  subspace  $\text{AdS}_{(3|2,0)}$ . Consider a Killing vector field of  $(3,0)$  AdS superspace,

$$\xi = \xi^a \mathcal{D}_a + \xi_{ij}^\alpha \mathcal{D}_\alpha^{ij} . \quad (6.8)$$

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<sup>17</sup>This is in fact a normal coordinate system for  $\text{AdS}_{(3|3,0)}$  around the submanifold  $\text{AdS}_{(3|2,0)}$ .



We recall that  $\xi$  obeys the Killing equations (5.4) which are equivalent to (5.5a) – (5.5c). We introduce  $\mathcal{N} = 2$  projections of the transformation parameters involved

$$\tau^a := \xi^a|, \quad \tau^\alpha := \xi_{11}^\alpha|, \quad \bar{\tau}^\alpha = \xi_{22}^\alpha|, \quad t := i\Lambda^{12}| = \bar{t}, \quad t^{ab} := \Lambda^{ab}|; \quad (6.9a)$$

$$\rho^\alpha := -i\xi_{12}^\alpha| = \bar{\rho}^\alpha, \quad \bar{\varepsilon} := \Lambda^{11}|, \quad \varepsilon = \Lambda^{22}| = \Lambda_{11}|. \quad (6.9b)$$

The important point is that the parameters  $(\tau^a, \tau^\alpha, \bar{\tau}_\alpha, t^{\alpha\beta}, t)$  describe the infinitesimal isometries of the (2,0) AdS superspace [13]. Such transformations are generated by the Killing vector fields,  $\tau = \tau^a \mathbf{D}_a + \tau^\alpha \mathbf{D}_\alpha + \bar{\tau}_\alpha \bar{\mathbf{D}}^\alpha$ , obeying the Killing equation

$$\left[ \tau + it\mathcal{J} + \frac{1}{2}t^{bc}\mathcal{M}_{bc}, \mathbf{D}_A \right] = 0, \quad (6.10)$$

for some parameters  $t$  and  $t^{ab}$ . This equation is equivalent to

$$4S\tau_\alpha = \bar{\mathbf{D}}_\alpha t = \frac{2i}{3}S\bar{\mathbf{D}}^\beta \tau_{\alpha\beta} = \frac{i}{3}\bar{\mathbf{D}}^\beta t_{\alpha\beta}, \quad (6.11a)$$

$$\bar{\mathbf{D}}_\alpha \tau_\beta = \mathbf{D}_{(\alpha} \tau_{\beta\gamma)} = \mathbf{D}_{(\alpha} t_{\beta\gamma)} = 0, \quad (6.11b)$$

$$\mathbf{D}_\gamma \tau^\gamma = -\bar{\mathbf{D}}^\gamma \bar{\tau}_\gamma = 2it, \quad (6.11c)$$

$$\mathbf{D}_{(\alpha} \tau_{\beta)} = -\bar{\mathbf{D}}_{(\alpha} \bar{\tau}_{\beta)} = \frac{1}{2}t_{\alpha\beta} + S\tau_{\alpha\beta}. \quad (6.11d)$$

These equations automatically follow from the (3,0) Killing equations, eqs. (5.5a) – (5.5c), upon  $\mathcal{N} = 2$  projection. The real parameter  $t|_{\theta=0} = \text{const}$  generates  $U(1)_R$  transformations of the (2,0) AdS superspace, where  $U(1)_R$  is a subgroup of the  $R$ -symmetry group  $SU(2)_R$  of the (3,0) AdS superspace.

The transformation parameters  $\rho^\alpha$ ,  $\varepsilon$  and  $\bar{\varepsilon}$  generate the third supersymmetry and those  $R$ -symmetry transformations which parametrise the coset  $SU(2)_R/U(1)_R$ . Making use of (5.6), one can show that  $\rho_\alpha$  is determined in terms of  $\varepsilon$  and  $\bar{\varepsilon}$ :

$$\rho_\alpha = -\frac{1}{8S}\mathbf{D}_\alpha \varepsilon = -\frac{1}{8S}\bar{\mathbf{D}}_\alpha \bar{\varepsilon}. \quad (6.12)$$

The parameters  $\varepsilon$  and  $\bar{\varepsilon}$  satisfy the following properties

$$\mathbf{D}_\alpha \varepsilon = \bar{\mathbf{D}}_\alpha \bar{\varepsilon}, \quad \bar{\mathbf{D}}_\alpha \varepsilon = 0, \quad \mathbf{D}_{\alpha\beta} \varepsilon = 0, \quad \mathbf{D}^2 \varepsilon = -8iS\bar{\varepsilon}. \quad (6.13)$$

These imply that the only independent components of  $\varepsilon$  are  $\varepsilon|_{\theta=0}$  and  $\mathbf{D}_\alpha \varepsilon|_{\theta=0}$ .

The notion of  $\mathcal{N} = 2$  projection is especially useful when dealing with projective multiplets. Given a covariant weight  $-n$  projective multiplet  $Q^{(n)}(v)$ , it can always be described in terms of a related superfield  $Q^{[n]}(\zeta)$  which depends on  $\zeta$  and is proportional

to the original superfield,  $Q^{[n]}(\zeta) \propto Q^{(n)}(v)$ . The precise definition of  $Q^{[n]}(\zeta)$  depends upon the specific projective multiplet under consideration. Using  $Q^{[n]}(\zeta)$ , the analyticity constraint (5.11) becomes

$$0 = \zeta^2 \mathcal{D}_\alpha^{11} Q^{[n]}(\zeta) - 2\zeta \mathcal{D}_\alpha^{12} Q^{[n]}(\zeta) + \mathcal{D}_\alpha^{22} Q^{[n]}(\zeta) , \quad (6.14)$$

or equivalently

$$\mathcal{D}_\alpha^{12} Q^{[n]}(\zeta) = \frac{1}{2} \left( \zeta \mathcal{D}_\alpha^{11} + \frac{1}{\zeta} \mathcal{D}_\alpha^{22} \right) Q^{[n]}(\zeta) . \quad (6.15)$$

This equation shows that the dependence of  $Q^{[n]}(x, \theta_{ij}, \zeta)$  on the Grassmann coordinates  $\theta_{12}^\mu$  is completely determined in terms of its dependence on the other Grassmann coordinates  $\theta_{11}^\mu$  and  $\theta_{22}^\mu$ . In other words, all information about the projective multiplet  $Q^{[n]}(\zeta)$  is encoded in its  $\mathcal{N} = 2$  projection  $Q^{[n]}(\zeta)|$ .

We now list the transformation laws of several projective multiplets under the (3,0) AdS isometry group,  $\text{OSp}(3|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$ . All multiplets will be projected to (2,0) AdS superspace, however will not indicate explicitly the bar-projection.

We recall that a weight- $n$  projective superfield  $Q^{(n)}$  transforms under the isometry group  $\text{OSp}(3|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$  as

$$\delta_\xi Q^{(n)}(z, v) = \left( \xi^a \mathcal{D}_a + \xi_{kl}^\alpha \mathcal{D}_\alpha^{kl} - \frac{1}{2} \Lambda^{(2)} \boldsymbol{\partial}^{(-2)} + \frac{n}{2} \Lambda^{(0)} \right) Q^{(n)}(z, v) , \quad (6.16)$$

which follows from eq. (5.9). Given an arctic weight- $n$  multiplet  $\Upsilon^{(n)}(v)$ , it can be conveniently represented as

$$\Upsilon^{(n)}(v) = (v^\perp)^n \Upsilon^{[n]}(\zeta) . \quad (6.17)$$

Then  $\Upsilon^{[n]}(\zeta)$  transforms as follows:

$$\delta_\xi \Upsilon^{[n]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \bar{\zeta}} - \frac{n}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \frac{\varepsilon}{\zeta} + \zeta \bar{\varepsilon} \right) \zeta \frac{\partial}{\partial \zeta} - \frac{n}{2} \zeta \bar{\varepsilon} \right\} \Upsilon^{[n]} . \quad (6.18)$$

Given an antarctic weight- $n$  multiplet  $\check{\Upsilon}^{(n)}(v)$ , it is represented in the form

$$\check{\Upsilon}^{(n)}(v) = (v^\perp)^n \check{\Upsilon}^{[n]}(\zeta) = (v^\perp)^n \zeta^n \check{\Upsilon}^{[n]}(\zeta) . \quad (6.19)$$

The transformation law of  $\check{\Upsilon}^{[n]}(\zeta)$  is

$$\delta_\xi \check{\Upsilon}^{[n]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \bar{\zeta}} + \frac{n}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \frac{\varepsilon}{\zeta} + \zeta \bar{\varepsilon} \right) \zeta \frac{\partial}{\partial \zeta} + \frac{n}{2\zeta} \varepsilon \right\} \check{\Upsilon}^{[n]} . \quad (6.20)$$

Given a real weight- $(2n)$  multiplet  $G^{(2n)}(v)$ ,  $\check{G}^{(2n)} = G^{(2n)}$ , it is represented as

$$G^{(2n)}(v) = (iv^\perp v^2)^n G^{[2n]}(\zeta) = (v^\perp)^{2n} (i\zeta)^n G^{[2n]}(\zeta) . \quad (6.21)$$

The transformation law of  $G^{[2n]}(\zeta)$  is

$$\delta_\xi G^{[2n]} = \left\{ \tau + it\zeta \frac{\partial}{\partial \zeta} + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \frac{\varepsilon}{\zeta} + \zeta \bar{\varepsilon} \right) \zeta \frac{\partial}{\partial \zeta} + \frac{n}{2} \left( \frac{\varepsilon}{\zeta} - \zeta \bar{\varepsilon} \right) \right\} G^{[2n]} . \quad (6.22)$$

To conclude the analysis of this subsection, we present the  $(3,0)$  supersymmetric action reduced to  $(2,0)$  superspace. In accordance with (6.21), associated with the Lagrangian  $\mathcal{L}^{(2)}(v)$  is the superfield  $\mathcal{L}^{[2]}(\zeta)$  defined by the rule  $\mathcal{L}^{(2)}(v) = i(v^\perp)^2 \zeta \mathcal{L}^{[2]}(\zeta)$ . It turns out that the  $(3,0)$  supersymmetric action (5.24) takes the following form in  $(2,0)$  AdS superspace

$$S[\mathcal{L}^{(2)}] = \oint_\gamma \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathbf{E} \mathcal{L}^{[2]} , \quad \mathbf{E}^{-1} := \text{Ber}(\mathbf{E}_A{}^M) . \quad (6.23)$$

To prove that (6.23) is the  $(2,0)$  reduction of the  $(3,0)$  action (5.24) we check explicitly that it is invariant under the full isometry group of  $(3,0)$  AdS superspace,  $\text{OSp}(3|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$ . Making use of (6.22), the variation of (6.23) is

$$\begin{aligned} \delta_\xi S[\mathcal{L}^{(2)}] = \oint_\gamma \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathbf{E} \left[ \tau + it\zeta \frac{\partial}{\partial \zeta} \right. \\ \left. + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha - \frac{1}{2} \zeta \bar{\varepsilon} + \frac{1}{2\zeta} \varepsilon + \left( \frac{1}{2} \zeta \bar{\varepsilon} + \frac{1}{2\zeta} \varepsilon \right) \zeta \frac{\partial}{\partial \zeta} \right] \mathcal{L}^{[2]} . \end{aligned} \quad (6.24)$$

The expression in the first line corresponds to the variation of  $\mathcal{L}^{[2]}$  under an infinitesimal isometry transformation of  $(2,0)$  AdS superspace. Since the action is manifestly invariant under the  $(2,0)$  AdS isometry group, this variation vanishes. For the remaining variation, upon integration by parts, we obtain

$$\delta_\xi S[\mathcal{L}^{(2)}] = \oint_\gamma \frac{d\zeta}{2\pi i} \int d^3x d^2\theta d^2\bar{\theta} \mathbf{E} \left( (i(\mathbf{D}_\alpha \rho^\alpha) - \bar{\varepsilon}) + \frac{1}{\zeta^2} (i(\bar{\mathbf{D}}^\alpha \rho_\alpha) + \varepsilon) \right) \mathcal{L}^{[2]} = 0 , \quad (6.25)$$

which is identically zero due to the identities

$$i\mathbf{D}_\alpha \rho_\beta = -\frac{1}{2} \varepsilon_{\alpha\beta} \bar{\varepsilon} , \quad i\bar{\mathbf{D}}_\alpha \rho_\beta = -\frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon . \quad (6.26)$$

We conclude by noticing that the auxiliary superfield  $\mathcal{C}^{(-4)}$  (5.24) has dropped out upon reduction to  $(2,0)$  AdS superspace.

## 6.2 AdS superspace reduction: (2,1) to (2,0)

We now turn to developing  $\mathcal{N} = 2$  reduction schemes for the projective multiplets in (2,1) AdS superspace. The main difference between the (3,0) and (2,1) AdS superspaces is that the latter possesses the covariantly constant tensor  $w^{ij}$  (which can be interpreted as the field strength of a frozen  $\mathcal{N} = 3$  vector multiplet). As follows from the algebra of (2,1) AdS covariant derivatives, eq. (4.16), the  $R$ -symmetry part of the holonomy group of this superspace is no longer  $SU(2)_R$ , as in the (3,0) case; instead it is the group  $U(1)_R$  which is associated with the generator by  $\mathcal{J} = -\frac{i}{2}w^{ij}\mathcal{J}_{ij}$ . Therefore, the local  $SU(2)_R$  group can be used to choose the  $SU(2)_R$  connection to be

$$\Phi_A^{kl} = w^{kl}\Phi_A . \quad (6.27)$$

In this gauge the tensor  $w^{ij}$  becomes strictly constant,  $w^{ij} = \text{const}$ , and turn into an invariant tensor of the (2,1) AdS isometry group  $OSp(2|2; \mathbb{R}) \times OSp(1|2, \mathbb{R})$ . It turns out that different numeric choices for  $w^{ij}$  correspond to the possibility to perform reduction either to the (2,0) AdS superspace or to the (1,1) one. In other words, the (2,1) AdS superspace allows two inequivalent  $\mathcal{N} = 2$  reduction schemes.

Here we focus on the AdS reduction  $(2,1) \rightarrow (2,0)$ . If we choose

$$w^{11} = w^{22} = 0 , \quad w^{12} = -w_{12} = -i \quad (6.28)$$

in the (2,1) algebra (4.16), then the operators  $\mathcal{D}_\alpha^{11}$  and  $(-\mathcal{D}_\alpha^{22})$  can be seen to satisfy the same (anti) commutation relations as eqs. (6.1a)–(6.1c) which are equivalent to the (2,0) AdS algebra (4.5a)–(4.5b). Therefore, the projection from (2,1) to (2,0) AdS formally proceeds exactly as in the (3,0), see the analysis around the equations (6.3)–(6.5). The only difference is that in (6.3) and (6.4) the (2,1) connections should be as in (6.27).

Consider the Killing vector fields,  $\xi^A = (\xi^a, \xi_{ij}^\alpha)$ , of the (2,1) AdS superspace. They obey the Killing equations, eq. (5.4), and hence

$$\mathcal{D}_\alpha^{ij}\xi_\beta^{kl} = \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}w^{jl} + \varepsilon^{jl}w^{ik})\Lambda - S(\varepsilon^{i(k}\varepsilon^{l)j} + w^{ij}w^{kl})\xi_{\alpha\beta} - \frac{1}{2}\varepsilon^{i(k}\varepsilon^{l)j}\Lambda_{\alpha\beta} , \quad (6.29a)$$

$$0 = \mathcal{D}_\gamma^{ij}\xi^{\alpha\gamma} + 6i\xi^{\alpha ij} , \quad 0 = \mathcal{D}_\gamma^{ij}\Lambda^{\alpha\gamma} - 12iS\xi_{kl}^\alpha(\varepsilon^{i(k}\varepsilon^{l)j} + w^{ij}w^{kl}) , \quad (6.29b)$$

$$0 = \mathcal{D}_{(\alpha}^{ij}\xi_{\beta\gamma)} = \mathcal{D}_{(\alpha}^{ij}\Lambda_{\beta\gamma)} , \quad (6.29c)$$

where we have used the fact that the  $R$ -symmetry group in the AdS (2,1) case reduces to  $U(1)_R$  and the corresponding transformation parameter is

$$\Lambda^{ij} = w^{ij}\Lambda , \quad \Lambda = \frac{1}{2}w_{kl}\Lambda^{kl} . \quad (6.30)$$

We now project the transformation parameters to (2,0) superspace

$$\tau^a := \xi^a|, \quad \tau^\alpha := \xi_{11}^\alpha|, \quad \bar{\tau}^\alpha := \xi_{22}^\alpha|, \quad t := i\Lambda^{12}| = \Lambda| = \bar{t}, \quad t^{ab} := \Lambda^{ab}|; \quad (6.31a)$$

$$\rho^\alpha := -i\xi_{12}^\alpha| = \bar{\rho}^\alpha. \quad (6.31b)$$

Because of (6.28), it holds that  $\Lambda^{11} = \Lambda^{22} = 0$ , which is clearly different from the (3,0) case. As in the (3,0) case, the parameters  $(\tau^a, \tau^\alpha, \bar{\tau}_\alpha, t^{\alpha\beta}, t)$  describe the infinitesimal isometries of the (2,0) AdS superspace. We recall that such transformations are generated by the Killing vector fields,

$$\tau = \tau^a \mathbf{D}_a + \tau^\alpha \mathbf{D}_\alpha + \bar{\tau}_\alpha \bar{\mathbf{D}}^\alpha,$$

obeying the equations (6.10) or, equivalently, (6.11). The real spinor parameter  $\rho^\alpha$  generates the third supersymmetry transformation. Making use of eqs. (6.29a)–(6.29c) gives

$$\mathbf{D}_\alpha \rho_\beta = \bar{\mathbf{D}}_\alpha \rho_\beta = 0. \quad (6.32)$$

These conditions mean that  $\rho_\alpha$  is an ordinary Killing spinor,

$$\mathbf{D}_{\beta\gamma} \rho_\alpha = S(\varepsilon_{\alpha\beta} \rho_\gamma + \varepsilon_{\alpha\gamma} \rho_\beta). \quad (6.33)$$

To complete the AdS superspace reduction  $(2,1) \rightarrow (2,0)$ , it remains to work out the transformation laws of projective multiplets under the (2,1) AdS isometry group,  $\text{OSp}(2|2; \mathbb{R}) \times \text{OSp}(1|2, \mathbb{R})$ . In (2,1) AdS superspace, a covariant weight- $n$  projective multiplet  $Q^{(n)}$  transforms as

$$\delta_\xi Q^{(n)}(z, v) = \left( \xi^a \mathcal{D}_a + \xi_{kl}^\alpha \mathcal{D}_\alpha^{kl} - \frac{1}{2} w^{(2)} \Lambda \boldsymbol{\partial}^{(-2)} + \frac{n}{2} w^{(0)} \Lambda \right) Q^{(n)}(z, v), \quad (6.34)$$

in accordance with eq. (5.9). We project this transformation law to (2,0) AdS superspace. Given an arctic weight- $n$  multiplet  $\Upsilon^{(n)}(v)$ , we associated with it the superfield  $\Upsilon^{[n]}(\zeta)$  defined by (6.17). The latter transforms as follows:

$$\delta_\xi \Upsilon^{[n]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \zeta} - \frac{n}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha \right\} \Upsilon^{[n]}. \quad (6.35)$$

Given an antarctic weight- $n$  multiplet  $\check{\Upsilon}^{(n)}(v)$ , we associated with it the superfield  $\check{\Upsilon}^{[n]}(\zeta)$  defined by (6.19). The latter transforms as follows:

$$\delta_\xi \check{\Upsilon}^{[n]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \zeta} + \frac{n}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha \right\} \check{\Upsilon}^{[n]}. \quad (6.36)$$

Given a real weight- $(2n)$  multiplet  $G^{(2n)}(v)$ ,  $\check{G}^{(2n)} = G^{(2n)}$ , we associate with it the superfield  $G^{[2n]}(\zeta)$ , eq. (6.21), with the transformation law

$$\delta_\xi G^{[2n]} = \left\{ \tau + it \zeta \frac{\partial}{\partial \zeta} + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha \right\} G^{[2n]} . \quad (6.37)$$

To conclude the subsection we note that the  $(2,1)$  AdS supersymmetric action reduced to  $(2,0)$  AdS superspace has exactly the same form as the  $(3,0)$  case: eq. (6.23). The proof that the action of the form (6.23) is invariant under the  $(2,1)$  isometries reduced to  $(2,0)$ , up to minor differences, goes along the same line of the  $(3,0)$  case.

### 6.3 AdS superspace reduction: $(2,1)$ to $(1,1)$

AdS superspace reduction  $(2,1) \rightarrow (1,1)$  corresponds to the following choice of  $w^{ij}$ :

$$w^{12} = 0 , \quad w := w^{11} , \quad \bar{w} = w^{22} = w_{\underline{1}\underline{1}} , \quad |w|^2 = 1 . \quad (6.38)$$

Making use of this  $w^{ij}$  in the (anti) commutation relations (4.16a)–(4.16c), and also introducing new AdS parameters

$$\mu = iS\bar{w}^2 , \quad \bar{\mu} = -iSw^2 , \quad (6.39)$$

we get the algebra

$$\{\mathcal{D}_\alpha^{11}, \mathcal{D}_\beta^{11}\} = -4\bar{\mu}\mathcal{M}_{\alpha\beta} , \quad \{(-\mathcal{D}_\alpha^{22}), (-\mathcal{D}_\beta^{22})\} = 4\mu\mathcal{M}_{\alpha\beta} , \quad (6.40a)$$

$$\{\mathcal{D}_\alpha^{11}, (-\mathcal{D}_\beta^{22})\} = -2i\mathcal{D}_{\alpha\beta} , \quad [\mathcal{D}_a, \mathcal{D}_\beta^{11}] = i\bar{\mu}(\gamma_a)_\beta{}^\gamma(-\mathcal{D}_\gamma^{22}) , \quad (6.40b)$$

$$[\mathcal{D}_a, (-\mathcal{D}_\beta^{22})] = -i\mu(\gamma_a)_\beta{}^\gamma\mathcal{D}_\gamma^{11} , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4|\mu|^2\mathcal{M}_{ab} . \quad (6.40c)$$

The (anti) commutation relations coincide with those corresponding to the covariant derivatives of  $(1,1)$  AdS superspace, eqs. (4.6a)–(4.6b). Since no  $U(1)_R$  curvature is present in the relations (6.40a)–(6.40c), we can use the local  $U(1)_R$  symmetry to choose a gauge in which the covariant derivatives  $\mathcal{D}_a, \mathcal{D}_\alpha^{11}$  and  $\mathcal{D}_\alpha^{22}$  have no  $U(1)_R$  connection.

The AdS superspace projection  $(2,1) \rightarrow (1,1)$  formally proceeds exactly as in the  $(3,0)$ , eqs. (6.3)–(6.5) with few differences:

- (i) the connection  $\Phi_A{}^{kl}$  in (6.4) should be as in (6.27);
- (ii) the general coordinate invariance can be used to choose a gauge

$$\mathcal{D}_\alpha^{11}| := \nabla_\alpha , \quad -\mathcal{D}_\alpha^{22}| := \bar{\nabla}_\alpha , \quad (6.41)$$

where

$$\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^\alpha) = \mathcal{E}_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} \mathcal{M}_{cd} \quad (6.42)$$

are the covariant derivatives for (1,1) anti-de Sitter superspace, which obey the (anti) commutation relations (4.6a)–(4.6b).

Consider the Killing vector fields,  $\xi^A = (\xi^a, \xi_{ij}^\alpha)$ , of the (2,1) AdS superspace. They obey the Killing equations (5.4) in which  $\Lambda^{ij}$  should be chosen in the form  $\Lambda^{ij} = w^{ij} \Lambda$  with  $w^{ij}$  given by eq. (6.38). We project the transformation parameters to (1,1) AdS superspace:

$$l^a = \xi^a|, \quad l^\alpha := \xi_{11}^\alpha|, \quad \bar{l}^\alpha = \xi_{22}^\alpha|, \quad \lambda^{ab} := \Lambda^{ab}|; \quad (6.43a)$$

$$\rho^\alpha := -i\xi_{12}^\alpha| = \bar{\rho}^\alpha, \quad \varepsilon := \Lambda| = \bar{\varepsilon}. \quad (6.43b)$$

The superfields  $(l^a, l^\alpha, \bar{l}_\alpha, \lambda^{ab})$  describe an infinitesimal isometry transformation of the (1,1) AdS superspace [13]. The isometries are generated by (1,1) AdS Killing vector fields,

$$l = l^a \nabla_a + l^\alpha \nabla_\alpha + \bar{l}_\alpha \bar{\nabla}^\alpha, \quad (6.44)$$

which are defined to obey the equations

$$\left[ l + \frac{1}{2} \lambda^{ab} \mathcal{M}_{ab}, \nabla_C \right] = 0, \quad (6.45)$$

which are equivalent to

$$0 = \nabla_{(\alpha} l_{\beta)} - \frac{1}{2} \lambda_{\alpha\beta}, \quad 0 = \bar{\nabla}_{(\alpha} l_{\beta)} + i\mu l_{\alpha\beta}, \quad \nabla_\alpha l^\alpha = \bar{\nabla}^\alpha l_\alpha = 0, \quad (6.46a)$$

$$0 = \nabla^\beta \lambda_{\alpha\beta} - 12\bar{\mu} l_\alpha, \quad 0 = \bar{\nabla}^\beta l_{\alpha\beta} + 6i l_\alpha, \quad \nabla_{(\alpha} l_{\beta\gamma)} = \nabla_{(\alpha} \lambda_{\beta\gamma)} = 0. \quad (6.46b)$$

The (1,1) AdS Killing vector fields can be shown to generate the supergroup  $\text{OSp}(1|2; \mathbb{R}) \times \text{OSp}(1|2; \mathbb{R})$ . The relations (6.46) follow by projecting the (2,1) Killing vector equations, (6.29a)–(6.29c), to the (1,1) AdS superspace.

The parameters  $\rho^\alpha|_{\theta=0}$  and  $\varepsilon|_{\theta=0}$  generate the third supersymmetry and U(1) transformations respectively. By using (6.29a)–(6.29c), one can derive the following equations

$$i\nabla_\alpha \rho_\beta = -\frac{1}{2} \varepsilon_{\alpha\beta} w \varepsilon, \quad i\bar{\nabla}_\alpha \rho_\beta = -\frac{1}{2} \varepsilon_{\alpha\beta} \bar{w} \varepsilon. \quad (6.47)$$

It can be further shown that the spinor superfield  $\rho_\alpha$  is determined in terms of  $\varepsilon$  as

$$\rho_\alpha = -\frac{1}{4Sw} \nabla_\alpha \varepsilon = -\frac{1}{4S\bar{w}} \bar{\nabla}_\alpha \varepsilon, \quad (6.48)$$

where  $\varepsilon$  can be proven to satisfy the equations

$$\bar{w}\nabla_\alpha\varepsilon = w\bar{\nabla}_\alpha\varepsilon, \quad (\nabla^2 - 4\bar{\mu})\varepsilon = 0, \quad (\bar{\nabla}^2 - 4\mu)\varepsilon = 0, \quad (6.49a)$$

$$(i\nabla^\alpha\bar{\nabla}_\alpha - 4|\mu|)\varepsilon = 0, \quad \nabla_{(\alpha}\bar{\nabla}_{\beta)}\varepsilon = \nabla_{\alpha\beta}\varepsilon = 0. \quad (6.49b)$$

To complete the AdS superspace reduction  $(2,1) \rightarrow (2,0)$ , it remains to work out the transformation laws of projective multiplets under the  $(2,1)$  AdS isometry group,  $\text{OSp}(2|2;\mathbb{R}) \times \text{OSp}(1|2,\mathbb{R})$ . In  $(2,1)$  AdS superspace, a covariant weight- $n$  projective multiplet  $Q^{(n)}$  transforms as in (6.34). We project the transformation law (6.34) to the  $(1,1)$  AdS superspace. Given an arctic weight- $n$  multiplet  $\Upsilon^{(n)}(v)$ , we associated with it the superfield  $\Upsilon^{[n]}(\zeta)$  defined by (6.17). The latter transforms as follows:

$$\delta_\xi \Upsilon^{[n]} = \left\{ l + i\zeta\rho^\alpha\nabla_\alpha + \frac{i}{\zeta}\rho_\alpha\bar{\nabla}^\alpha + \frac{1}{2}\varepsilon\left(w\zeta + \frac{\bar{w}}{\zeta}\right)\zeta\frac{\partial}{\partial\zeta} - \frac{n}{2}\varepsilon w\zeta \right\} \Upsilon^{[n]}. \quad (6.50)$$

Given an antarctic weight- $n$  multiplet  $\check{\Upsilon}^{(n)}(v)$ , we associated with it the superfield  $\check{\Upsilon}^{[n]}(\zeta)$  defined by (6.19). The latter transforms as follows:

$$\delta_\xi \check{\Upsilon}^{[n]} = \left\{ l + i\zeta\rho^\alpha\nabla_\alpha + \frac{i}{\zeta}\rho_\alpha\bar{\nabla}^\alpha + \frac{1}{2}\varepsilon\left(w\zeta + \frac{\bar{w}}{\zeta}\right)\zeta\frac{\partial}{\partial\zeta} + \frac{n}{2}\varepsilon\frac{\bar{w}}{\zeta} \right\} \check{\Upsilon}^{[n]}. \quad (6.51)$$

Given a real weight- $(2n)$  multiplet  $G^{(2n)}(v)$ ,  $\check{G}^{(2n)} = G^{(2n)}$ , we associate with it the superfield  $G^{[2n]}(\zeta)$ , eq. (6.21), with the transformation law

$$\delta_\xi G^{[2n]} = \left\{ l + i\zeta\rho^\alpha\nabla_\alpha + \frac{i}{\zeta}\rho_\alpha\bar{\nabla}^\alpha + \frac{1}{2}\varepsilon\left(w\zeta + \frac{\bar{w}}{\zeta}\right)\zeta\frac{\partial}{\partial\zeta} + \frac{n}{2}\varepsilon\left(\frac{\bar{w}}{\zeta} - w\zeta\right) \right\} G^{[2n]}. \quad (6.52)$$

Now let us show that the  $(1,1)$  AdS supersymmetric action in  $(1,1)$  AdS superspace takes the form

$$S[\mathcal{L}^{(2)}] = \oint_\gamma \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \mathcal{L}^{[2]}, \quad \mathcal{E}^{-1} := \text{Ber}(\mathcal{E}_A^M), \quad (6.53)$$

where  $(x, \theta^\mu, \bar{\theta}_\mu)$  are the local coordinates on  $(1,1)$  AdS superspace, and  $\mathcal{E}_A^M$  is the vielbein, eq. (6.42). The Lagrangian  $\mathcal{L}^{[2]}$  is defined as usual,  $\mathcal{L}^{(2)}(v) = i(v^\perp)^2 \zeta \mathcal{L}^{[2]}(\zeta)$ . We have to demonstrate that the action is invariant under the  $(2,1)$  AdS isometry group  $\text{OSp}(2|2;\mathbb{R}) \times \text{OSp}(1|2,\mathbb{R})$ . By using the transformation law (6.52) for  $n = 1$ , the variation of (6.53) is

$$\begin{aligned} \delta_\xi S[\mathcal{L}^{[2]}] = & \oint_\gamma \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \left[ l \right. \\ & \left. + i\zeta\rho^\alpha\nabla_\alpha + \frac{i}{\zeta}\rho_\alpha\bar{\nabla}^\alpha - \frac{w}{2}\zeta\varepsilon + \frac{\bar{w}}{2\zeta}\varepsilon + \frac{1}{2}\left(\zeta w + \frac{\bar{w}}{\zeta}\right)\varepsilon\zeta\frac{\partial}{\partial\zeta} \right] \mathcal{L}^{[2]}. \end{aligned} \quad (6.54)$$



The variation in the first line does not contribute to  $\delta_\xi S[\mathcal{L}^{[2]}]$ , since the action is manifestly (1,1) AdS supersymmetric. Integrating by parts in the second line gives

$$\delta_\xi S[\mathcal{L}^{[2]}] = \oint_\gamma \frac{d\zeta}{2\pi i} \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \left( (i(\nabla_\alpha \rho^\alpha) - w\varepsilon) + \frac{1}{\zeta^2} (i(\bar{\nabla}^\alpha \rho_\alpha) + \bar{w}\varepsilon) \right) \mathcal{L}^{[2]} = 0, \quad (6.55)$$

which is identically zero due to (6.47).

## 7 $\mathcal{N} = 3$ supersymmetric sigma models in AdS

We are now prepared to apply the formalism developed above to construct general (3,0) and (2,1) supersymmetric nonlinear  $\sigma$ -models in  $\text{AdS}_3$ .

### 7.1 Sigma models with (3,0) AdS supersymmetry

By analogy with the rigid supersymmetric case, it is natural to expect that a general nonlinear  $\sigma$ -model with (3,0) AdS supersymmetry can be realised in terms of covariant weight-one arctic multiplets  $\Upsilon^{(1)I}(v)$  and their smile-conjugates  $\check{\Upsilon}^{(1)\bar{I}}(v)$ , with  $I = 1, \dots, n$ . What can be said about the Lagrangian  $\mathcal{L}^{(2)}$  of such a theory? The specific feature of the (3,0) AdS superspace is that there are no *background* projective multiplets which are invariant under the isometry supergroup  $\text{OSp}(3|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$ .<sup>18</sup> In order for  $\mathcal{L}^{(2)}$  to be a covariant weight-two projective multiplet, it cannot depend explicitly on  $v^i$ . It must be a function of the dynamical superfields only,

$$\mathcal{L}^{(2)} = i K(\Upsilon^{(1)}, \check{\Upsilon}^{(1)}), \quad (7.1)$$

where  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  is a homogeneous function of its arguments of degree one,

$$\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \bar{\Phi}^{\bar{I}} \frac{\partial}{\partial \bar{\Phi}^{\bar{I}}} \right) K(\Phi, \bar{\Phi}) = 2K(\Phi, \bar{\Phi}). \quad (7.2)$$

In order for  $\mathcal{L}^{(2)}$  to be real with respect to the smile-conjugation

$$\smile: \quad \Upsilon^{(1)} \rightarrow \check{\Upsilon}^{(1)}, \quad \check{\Upsilon}^{(1)} \rightarrow -\Upsilon^{(1)},$$

it suffices to subject  $K(\Phi, \bar{\Phi})$  to additional conditions [40, 41]

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}), \quad \bar{K} = K. \quad (7.3)$$

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<sup>18</sup>The situation is completely different in the (2,1) AdS case where the background  $\mathcal{O}(2)$  multiplet  $w^{(2)} := v_i v_j w^{ij}$  is invariant under the isometry group  $\text{OSp}(2|2; \mathbb{R}) \times \text{OSp}(1|2; \mathbb{R})$ .

This condition means that  $K(\Phi, \bar{\Phi})$  can be interpreted as the Kähler potential of a Kähler cone, see e.g. [50]. By definition, this is a Kähler manifold  $(\mathcal{M}, g_{I\bar{J}})$  possessing a homothetic conformal Killing vector  $\chi$

$$\chi = \chi^I \frac{\partial}{\partial \Phi^I} + \bar{\chi}^{\bar{I}} \frac{\partial}{\partial \bar{\Phi}^{\bar{I}}} \equiv \chi^\mu \frac{\partial}{\partial \varphi^\mu} , \quad (7.4)$$

with the property

$$\nabla_\nu \chi^\mu = \delta_\nu^\mu \iff \nabla_J \chi^I = \delta_J^I , \quad \nabla_{\bar{J}} \chi^I = \partial_{\bar{J}} \chi^I = 0 . \quad (7.5)$$

In particular,  $\chi$  is holomorphic. Its properties include:

$$g_{I\bar{J}} \chi^I \bar{\chi}^{\bar{J}} = K , \quad \chi_I := g_{I\bar{J}} \bar{\chi}^{\bar{J}} = \partial_I K \implies \chi^I K_I = K , \quad (7.6)$$

with  $K$  the Kähler potential. Local complex coordinates for  $\mathcal{M}$  can always be chosen such that

$$\chi = \Phi^I \frac{\partial}{\partial \Phi^I} + \bar{\Phi}^{\bar{I}} \frac{\partial}{\partial \bar{\Phi}^{\bar{I}}} , \quad (7.7)$$

which correspond to our specific case, eq. (7.3).

In 3D  $\mathcal{N} = 3$  flat projective superspace, any nonlinear  $\sigma$ -model with Lagrangian specified by eqs. (7.1) and (7.3) is  $\mathcal{N} = 3$  superconformal [12] (which is a generalisation of the earlier results in the 4D  $\mathcal{N} = 2$  case [40, 41]). The target spaces of these  $\sigma$ -models are hyperKähler cones, see e.g. [50, 51] and references therein. Since (3,0) AdS superspace is conformally related to  $\mathcal{N} = 3$  Minkowski superspace, we conclude that general nonlinear  $\sigma$ -models in (3,0) AdS superspace are  $\mathcal{N} = 3$  superconformal.

Consider the  $\sigma$ -model

$$S = \oint_\gamma \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathbf{E} \mathcal{L}^{[2]} , \quad (7.8)$$

where

$$\mathcal{L}^{[2]} := \frac{1}{\zeta} K(\Upsilon^{[1]}, \zeta \check{\Upsilon}^{[1]}) \quad (7.9)$$

At the moment we assume only the homogeneity condition (7.2). The transformation law of  $\mathcal{L}^{[2]}$  must be

$$\delta_\xi \mathcal{L}^{[2]} = \left\{ \tau + it \zeta \frac{\partial}{\partial \zeta} + i \zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \zeta \bar{\varepsilon} + \frac{1}{\zeta} \varepsilon \right) \zeta \frac{\partial}{\partial \zeta} - \frac{1}{2} \zeta \bar{\varepsilon} + \frac{1}{2\zeta} \varepsilon \right\} \mathcal{L}^{[2]} . \quad (7.10)$$

This should be induced by the variations of  $\Upsilon^{[1]}$  and  $\check{\Upsilon}^{[1]}$  in (7.9), which are

$$\delta_\xi \Upsilon^{[1]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \zeta} - \frac{1}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \zeta \bar{\varepsilon} + \frac{1}{\zeta} \varepsilon \right) \zeta \frac{\partial}{\partial \zeta} - \frac{1}{2} \zeta \bar{\varepsilon} \right\} \Upsilon^{[1]}, \quad (7.11a)$$

$$\delta_\xi \check{\Upsilon}^{[1]} = \left\{ \tau + it \left( \zeta \frac{\partial}{\partial \zeta} + \frac{1}{2} \right) + i\zeta \rho^\alpha \mathbf{D}_\alpha + \frac{i}{\zeta} \rho_\alpha \bar{\mathbf{D}}^\alpha + \frac{1}{2} \left( \zeta \bar{\varepsilon} + \frac{1}{\zeta} \varepsilon \right) \zeta \frac{\partial}{\partial \zeta} + \frac{1}{2\zeta} \varepsilon \right\} \check{\Upsilon}^{[1]}. \quad (7.11b)$$

It is a short calculation to show that  $\mathcal{L}^{[2]}$  given by eq. (7.9) transforms as in (7.10) if eq. (7.2) holds. On the other hand, the Lagrangian (7.9) is real under the smile conjugation provided the stronger conditions (7.3) hold.

## 7.2 Sigma models with (2,1) AdS supersymmetry

Unlike the (3,0) AdS superspace studied above, the (2,1) AdS superspace possesses a nontrivial covariantly constant tensor – the  $\mathcal{O}(2)$  multiplet  $w^{(2)} = v_i v_j w^{ij}$ , with  $w^{ij}$  the parameter of the (2,1) AdS algebra (4.16a)–(4.16c). This invariant tensor can be used to construct supersymmetric theories generated by Lagrangians of the form<sup>19</sup>

$$\mathcal{L}^{(2)} = w^{(2)} \mathcal{L}^{(0)}, \quad (7.12)$$

for some covariant real weight-zero projective multiplet  $\mathcal{L}^{(0)}$ .

In (2,1) AdS superspace, general nonlinear  $\sigma$ -models can be described in terms of covariant weight-zero arctic multiplets  $\Upsilon^I(v)$  and their smile-conjugates  $\check{\Upsilon}^{\bar{I}}(v)$  using the Lagrangian

$$\mathcal{L}^{(2)} = w^{(2)} K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}), \quad (7.13)$$

where  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  is the Kähler potential of a real analytic Kähler manifold  $\mathcal{X}$ . The interpretation of  $K$  as a Kähler potential is consistent, since the action generated by (7.13) turns out to be invariant under Kähler transformations of the form

$$K(\Upsilon, \check{\Upsilon}) \rightarrow K(\Upsilon, \check{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\check{\Upsilon}), \quad (7.14)$$

with  $\Lambda(\Phi^I)$  a holomorphic function. The target space  $\mathcal{M}$  of this  $\sigma$ -model proves to be an open domain of the zero section of the cotangent bundle of  $\mathcal{X}$ ,  $\mathcal{M} \subset T^* \mathcal{X}$ . This can be shown by generalizing the flat-superspace considerations of [36, 52].

In general,  $K(\Phi, \bar{\Phi})$  in (7.13) is an arbitrary real analytic function of  $n$  complex variables. In the case that  $K(\Phi, \bar{\Phi})$  obeys the homogeneity condition (7.3), the Lagrangian

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<sup>19</sup>Similar models exist in 5D  $\mathcal{N} = 1$  AdS [49] and 4D  $\mathcal{N} = 2$  AdS [29].

(7.13) proves to define an  $\mathcal{N} = 3$  superconformal  $\sigma$ -model. Such a theory can be reformulated entirely in terms of covariant weight-one arctic multiplets,  $\Upsilon^{(1)I}(v)$ , and their smile-conjugates in complete analogy with the four-dimensional  $\mathcal{N} = 2$  supersymmetric  $\sigma$ -models in AdS [22]. This requires to make use of an intrinsic hypermultiplet,  $q^i$ , associated with the (2,1) AdS superspace. This hypermultiplet is defined in complete analogy with the 4D consideration given in section 2.2 of [22].

It has been shown in the previous section that the (2,1) AdS superspace allows two types of  $\mathcal{N} = 2$  reduction, depending on the choice of  $w^{ij}$  made. Any field theory in  $\text{AdS}_{(3|2,1)}$  can be reformulated as a dynamical system in  $\text{AdS}_{(3|2,0)}$  or in  $\text{AdS}_{(3|1,1)}$ . Upon reduction to the (2,0) AdS superspace, the supersymmetric  $\sigma$ -model (7.13) proves to be described by the action

$$S = \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathbf{E} K(\Upsilon, \check{\Upsilon}) , \quad (7.15)$$

where the dynamical variables  $\Upsilon^I$  and their smile-conjugates  $\check{\Upsilon}^{\bar{I}}$  have the form

$$\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I = \Phi^I + \zeta \Sigma^I + \dots , \quad \check{\Upsilon}^{\bar{I}}(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}_n^{\bar{I}} . \quad (7.16)$$

Here  $\Phi^I := \Upsilon_0^I$  and  $\Sigma^I := \Upsilon_1^I$  are covariantly chiral and complex linear superfields, respectively,

$$\bar{\mathbf{D}}_{\alpha} \Phi^I = 0 , \quad \bar{\mathbf{D}}^2 \Sigma^I = 0 , \quad (7.17)$$

while the other components  $\Upsilon_2^I, \Upsilon_3^I, \dots$ , are unconstrained complex  $\mathcal{N} = 2$  superfields.

It is known that (2,0) AdS supersymmetry allows only  $R$ -invariant  $\sigma$ -model couplings [13]. As concerns the (2,1) supersymmetric  $\sigma$ -model (7.15), it possesses the following  $U(1)$  symmetry:

$$\Upsilon(\zeta) \rightarrow \Upsilon(e^{i\alpha}\zeta) , \quad \alpha \in \mathbb{R} , \quad (7.18)$$

compare with [53]. This symmetry is a special case of the transformation law (6.35) obtained by setting  $t = \alpha = \text{const}$  and switching off the other parameters.

Upon reduction to the (1,1) AdS superspace, the supersymmetric  $\sigma$ -model (7.13) is described by the action

$$S = \frac{1}{2} \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} w^{[2]} K(\Upsilon, \check{\Upsilon}) , \quad w^{[2]} = -i \left( \frac{\bar{w}}{\zeta} + w\zeta \right) . \quad (7.19)$$

The dynamical variables  $\Upsilon^I(\zeta)$  and  $\check{\Upsilon}^{\bar{I}}(\zeta)$  have the functional form (7.16) where  $\Phi^I$  and  $\Sigma^I$  obey the constraints

$$\bar{\nabla}_\alpha \Phi^I = 0 \ , \quad (\bar{\nabla}^2 - 4\mu)\Sigma^I = 0 \ , \quad (7.20)$$

and the other components  $\Upsilon_2^I, \Upsilon_3^I, \dots$ , are unconstrained complex  $\mathcal{N} = 2$  superfields.

## 8 Conclusion

In conclusion, we briefly summarise our main results and list some open problems. In this paper we introduced the three-dimensional  $(p, q)$  AdS superspaces, studied their geometric properties and proved their conformal flatness when  $X^{IJKL} = 0$ . Building on the results of [6], we then developed the fully-fledged projective-superspace formalism to construct off-shell  $\mathcal{N} = 3$  rigid supersymmetric field theories in  $\text{AdS}_3$ . There are two types of such theories, with (3,0) and (2,1) AdS supersymmetry respectively. We are especially interested in theories possessing  $(p, q)$  AdS supersymmetry with  $\mathcal{N} = p + q \leq 4$  because *nonlinear*  $\sigma$ -models exist only in these cases. We recall that the  $\sigma$ -models with  $\mathcal{N} = p + q = 2$  were studied earlier in [14, 15, 13]. The explicit construction of  $(p, q)$  supersymmetric  $\sigma$ -models with  $p + q = 3$  was the subject of the present work. An open interesting problem is to extend our analysis given in this paper to the cases  $\mathcal{N} = p + q = 4$ . Conceptually, this should be similar to the  $\mathcal{N} = 3$  case studied above, however some nontrivial new aspects will emerge. In particular, of special interest are those (4,0) supersymmetric  $\sigma$ -models which correspond to the extremal case (4.21).

In this paper we constructed the general (3,0) and (2,1) supersymmetric  $\sigma$ -models described by off-shell polar hypermultiplets defined on the (3,0) and (2,1) AdS superspaces respectively. We then reduced these  $\sigma$ -models to certain  $\mathcal{N} = 2$  AdS superspaces. An interesting open problem is to reformulate the  $\sigma$ -models obtained in terms of  $\mathcal{N} = 2$  chiral superfields in  $\text{AdS}_3$ . (The importance of such a formulation is that it should provide a direct access to the hyperkähler geometry of the target space [54, 19, 20].) This can be achieved by generalising the approaches developed in [55, 56, 22].

As shown in this paper, the  $\mathcal{N} = 3$  AdS supersymmetry imposes nontrivial restrictions on the  $\sigma$ -model hyperkähler target spaces. The most unexpected outcome is that (3,0) AdS supersymmetry requires the  $\sigma$ -model target spaces to be hyperkähler cones. Nevertheless, this result has a natural geometric origin. The main difference between the two types of  $\mathcal{N} = 3$  supersymmetric  $\sigma$ -models in  $\text{AdS}_3$  is encoded in the corresponding

$R$ -symmetry groups:  $SO(3)$  in the  $(3,0)$  case and  $SO(2)$  in the  $(2,1)$  case. It can be shown that any one-dimensional subgroup  $H = SO(2)$  of the  $R$ -symmetry group acts faithfully by rotations on the two-sphere of complex structures of the hyperkähler target space  $(\mathcal{M}, g, \mathcal{J}_A)$ .<sup>20</sup> Here  $g_{\mu\nu}$  is the hyperkähler metric, and  $(\mathcal{J}_A)^\mu{}_\nu$  is the complex structures of  $\mathcal{M}$ ,  $\mathcal{J}_A = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ , obeying the quaternionic algebra  $\mathcal{J}_A \mathcal{J}_B = -\delta_{AB} \mathbb{1} + \varepsilon_{ABC} \mathcal{J}_C$ . Suppose that  $\mathcal{J}_3$  is invariant under the action of the subgroup  $H$ , and let  $V^\mu$  be the Killing vector  $V^\mu$  associated with  $H$ . Without loss of generality, we have

$$\mathcal{L}_V \mathcal{J}_1 = -\mathcal{J}_2, \quad \mathcal{L}_V \mathcal{J}_2 = +\mathcal{J}_1, \quad \mathcal{L}_V \mathcal{J}_3 = 0. \quad (8.1)$$

The Killing vector  $V^\mu$  is holomorphic with respect to  $\mathcal{J}_3$ , and we can introduce the corresponding Killing potential  $\mathcal{K}$  defined by

$$V^\mu = \frac{1}{2} (\mathcal{J}_3)^\mu{}_\nu \nabla^\nu \mathcal{K}. \quad (8.2)$$

As shown in [22],  $\mathcal{K}$  is a globally defined function over  $\mathcal{M}$ , and is the Kähler potential with respect to  $\mathcal{J}_1$  and  $\mathcal{J}_2$  and indeed any complex structure  $\mathcal{J}_\perp$  which is perpendicular to  $\mathcal{J}_3$ . In other words,

$$g_{\mu\nu} = \frac{1}{2} \nabla_\mu \nabla_\nu \mathcal{K} + \frac{1}{2} (\mathcal{J}_\perp)_\mu{}^\rho (\mathcal{J}_\perp)_\nu{}^\sigma \nabla_\rho \nabla_\sigma \mathcal{K}. \quad (8.3)$$

It follows that the Kähler forms associated with  $\mathcal{J}_1$  and  $\mathcal{J}_2$  are exact, and thus  $\mathcal{M}$  is non-compact [19, 20, 22]. As shown in [22], the Kähler potential  $\mathbb{K}$  with respect to  $\mathcal{J}_3$  can be chosen such that

$$(\mathcal{J}_3)^\mu{}_\nu V^\nu \mathbb{K}_\mu = -\mathcal{K}. \quad (8.4)$$

So far, we have taken into account only the fact that the  $R$ -symmetry group contain a subgroup  $SO(2)$ . In the case that the  $R$ -symmetry group coincides with  $SO(3)$ , the above consideration implies that  $\mathbb{K} = \mathcal{K}$ , and hence

$$\nabla^\mu \mathcal{K} \nabla_\mu \mathcal{K} = 2\mathcal{K}. \quad (8.5)$$

We further deduce that  $\chi^\mu = g^{\mu\nu} \nabla_\nu \mathcal{K}$  is a homothetic conformal Killing vector,

$$\nabla_\nu \chi^\mu = \delta_\nu^\mu, \quad (8.6)$$

and therefore  $\mathcal{M}$  is a hyperkähler cone [50, 51]. In regard to the above discussion, we should also mention an interesting work [58] in which it was shown that a sufficient

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<sup>20</sup>The existence of such hyperkähler spaces was pointed out twenty five years ago in [57].

condition for a 4D  $\mathcal{N} = 2$   $\sigma$ -model in projective superspace to be superconformal is that its  $R$ -symmetry is  $SO(3)$ .

The supergravity techniques of [6] can straightforwardly be applied to construct off-shell  $\sigma$ -models in the deformed  $\mathcal{N} = 4$  Minkowski superspace described by covariant derivatives obeying the (anti) commutation relations

$$\{\mathcal{D}_{\alpha}^{\bar{i}\bar{i}}, \mathcal{D}_{\beta}^{j\bar{j}}\} = 2i\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}X\mathbf{L}^{ij} - 2i\varepsilon_{\alpha\beta}\varepsilon^{ij}X\mathbf{R}^{\bar{i}\bar{j}}, \quad (8.7a)$$

$$[\mathcal{D}_a, \mathcal{D}_{\beta}^{j\bar{j}}] = 0, \quad [\mathcal{D}_a, \mathcal{D}_b] = 0 \quad (8.7b)$$

which follow from (4.20) by setting  $S = 0$ . An interesting open problem is to understand the target space geometry of such  $\mathcal{N} = 4$  supersymmetric nonlinear  $\sigma$ -models.

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## A Derivation of (5.37)

Here we sketch the derivation of the action (5.37) by requiring its invariance under the projective transformations (5.29). The derivation is actually similar to those given in [49, 48, 42], and the interested reader is referred to those papers for more technical details.

The strategy is to start from the zero-order term  $S_0$  in (5.36), vary it under the infinitesimal transformation (5.29) and add iteratively extra terms to the action, which cancel the variation order by order, such that the final action is invariant. Instead of

working with the general infinitesimal transformation (5.29), it suffices to deal with the  $b$  variation

$$\delta u_i = b v_i , \quad (\text{A.8})$$

since the  $a$  and  $c$  variations do not contribute if degrees of homogeneity in  $v$  and  $u$  are chosen properly. The transformation (A.8) induces the following variations:

$$\delta \mathcal{D}_\alpha^{(-2)} = \frac{2b}{(v, u)} \mathcal{D}_\alpha^{(0)} , \quad \delta \mathcal{D}_\alpha^{(0)} = \frac{b}{(v, u)} \mathcal{D}_\alpha^{(2)} , \quad (\text{A.9a})$$

$$\delta \mathcal{S}^{(-4)} = \frac{4b}{(v, u)} \mathcal{S}^{(-2)} , \quad \delta \mathcal{S}^{(-2)} = \frac{3b}{(v, u)} \mathcal{S}^{(0)} , \quad \delta \mathcal{S}^{(0)} = \frac{2b}{(v, u)} \mathcal{S}^{(2)} , \quad (\text{A.9b})$$

where  $\mathcal{S}^{(2)} := (v_i v_j v_k u_l \mathcal{S}^{ijkl}) / (v, u)$ . Let us compute the variation of  $S_0$  defined by (5.36). Making use of (A.9a)–(A.9b) and the analyticity condition  $\mathcal{D}_\alpha^{(2)} \mathcal{L}^{(2)} = 0$  gives

$$\begin{aligned} \delta S_0 = \frac{1}{8\pi} \int d^3x e \oint_\gamma v_i dv^i \frac{b}{(v, u)} & \left[ 2 \{ \mathcal{D}^{(0)\alpha}, \mathcal{D}_\alpha^{(-2)} \} (\mathcal{D}^{(0)})^2 + 4 \mathcal{D}^{(-2)\alpha} \mathcal{D}_\alpha^{(0)} (\mathcal{D}^{(0)})^2 \right. \\ & \left. + (\mathcal{D}^{(-2)})^2 \{ \mathcal{D}^{(2)\alpha}, \mathcal{D}_\alpha^{(0)} \} \right] \mathcal{L}^{(2)} || . \end{aligned} \quad (\text{A.10})$$

The integrand can be considerably simplified. Using the algebra of covariant derivatives, (5.1a)–(5.1b), it is not difficult to derive the following relation

$$\mathcal{D}_\alpha^{(0)} (\mathcal{D}^{(0)})^2 \mathcal{L}^{(2)} = \left( i \mathcal{D}_{\alpha\beta} \mathcal{D}^{(0)\beta} + i(2\mathcal{S}^{(0)} - \mathcal{S}) \mathcal{D}_\alpha^{(0)} + i \mathcal{S}^{(2)} \mathcal{D}_\alpha^{(-2)} \right) \mathcal{L}^{(2)} , \quad (\text{A.11})$$

which has to be plugged in eq. (A.10). Next, we evaluate the anti-commutators in (A.10) and iteratively move all the Lorentz and SU(2) generators to the right. Once they hit  $\mathcal{L}^{(2)}$  we use the identities  $\mathcal{M}_{\alpha\beta} \mathcal{L}^{(2)} = v_i v_j \mathcal{J}^{ij} \mathcal{L}^{(2)} = 0$  and  $v_i u_j \mathcal{J}^{ij} \mathcal{L}^{(2)} = -(v, u) \mathcal{L}^{(2)}$ . To compute the contributions coming from  $u_i u_j \mathcal{J}^{ij} \mathcal{L}^{(2)}$  one has to use the following formula

$$\oint \frac{v_i dv^i}{(v, u)^6} b \mathcal{T}^{(3)} u_i u_j \mathcal{J}^{ij} \mathcal{L}^{(2)} = \oint \frac{v_i dv^i}{(v, u)^5} \left\{ b \left( u^k \frac{\partial}{\partial v^k} \mathcal{T}^{(3)} \right) \mathcal{L}^{(2)} \right\} . \quad (\text{A.12})$$

This can be obtained using the results of [42, 48], and it holds for any operator  $\mathcal{T}^{(3)}$  which is a function of  $v$  and  $u$  and homogeneous in  $v$  of degree three :  $\mathcal{T}^{(3)}(cv) = c^3 \mathcal{T}^{(3)}(v)$ . The next step is to simplify the expression (A.10) obtained by moving the vector derivative  $\mathcal{D}_{\alpha\beta}$  coming from (A.11) to the left, which gives a total derivative to be ignored. The final result is

$$\begin{aligned} \delta S_0 = \frac{1}{8\pi} \int d^3x e \oint_\gamma v_i dv^i \frac{b}{(v, u)} & \left[ 16i \mathcal{S}^{(2)} (\mathcal{D}^{(-2)})^2 - 4i(\mathcal{S}^{(0)} + 4\mathcal{S}) \mathcal{D}^{(-2)\alpha} \mathcal{D}_\alpha^{(0)} \right. \\ & \left. + 40i \mathcal{S}^{(-2)} (\mathcal{D}^{(0)})^2 + 96 \mathcal{S}^{(2)} \mathcal{S}^{(-4)} \right] \mathcal{L}^{(2)} || . \end{aligned} \quad (\text{A.13})$$



To cancel this variation, we consider an additional functional of the form

$$S_{\text{extra}} = \int d^3x e \oint_{\gamma} \frac{v_i dv^i}{8\pi} \left[ i(a_1 \mathcal{S}^{(0)} + a_2 \mathcal{S})(\mathcal{D}^{(-2)})^2 + a_3 i \mathcal{S}^{(-2)} \mathcal{D}^{(-2)\alpha} \mathcal{D}_{\alpha}^{(0)} + a_4 i \mathcal{S}^{(-4)} (\mathcal{D}^{(0)})^2 \right. \\ \left. + a_5 \mathcal{S}^{(-2)} \mathcal{S}^{(-2)} + a_6 \mathcal{S}^{(-4)} \mathcal{S}^{(0)} + a_7 \mathcal{S}^{(-4)} \mathcal{S} \right] \mathcal{L}^{(2)} \Big| . \quad (\text{A.14})$$

By using the procedure described for the computation of  $\delta S_0$ , we derive

$$\delta S_{\text{extra}} = \frac{1}{8\pi} \int d^3x e \oint_{\gamma} v_i dv^i \frac{b}{(v, u)} \left[ 2i a_1 \mathcal{S}^{(2)} (\mathcal{D}^{(-2)})^2 + 2i(a_3 + 2a_4) \mathcal{S}^{(-2)} (\mathcal{D}^{(0)})^2 \right. \\ \left. + i \left( (4a_1 + 3a_3) \mathcal{S}^{(0)} + 4a_2 \mathcal{S} \right) \mathcal{D}^{(-2)\alpha} \mathcal{D}_{\alpha}^{(0)} + (-12a_4 + 2a_6) \mathcal{S}^{(-4)} \mathcal{S}^{(2)} \right. \\ \left. + \mathcal{S}^{(-2)} \left( (-24a_1 + 6a_5 + 4a_6) \mathcal{S}^{(0)} + (16a_1 - 16a_2 + 4a_7) \mathcal{S} \right) \right] \mathcal{L}^{(2)} \Big| . \quad (\text{A.15})$$

Imposing the condition  $\delta S_0 + \delta S_{\text{extra}} = 0$  fixes the coefficients

$$a_1 = -8 , \quad a_2 = 4 , \quad a_3 = 12 , \quad a_4 = -16 , \quad a_5 = -144 , \quad a_6 = 64 , \quad a_7 = 48 . \quad (\text{A.16})$$

These results give the desired action (5.37).

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